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Minimal Signature in Lifting of Operators, II

TIBERIU CONSTANTINESCU AND AURELIAN GHEONDEA

Department of Mathematics, INCREST,
Bdul Păcii 220, 79622 Bucharest, Romania

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This paper deals with a problem of lifting of operators in Krein spaces, with minimal signature numbers of the defects.

There are obtained necessary and sufficient conditions for the existence of liftings and parametrization formulae for the set of all solutions. As applications there are treated a problem of selfadjoint extensions of symmetric operators and a variant of lifting of commutants. © 1992 Academic Press, Inc.

1. INTRODUCTION

This work represents a continuation of our investigations [3, 10, 11], dealing with lifting of operators acting in spaces with indefinite metric and with control over the negative signature of the defects. This problem is connected with other works contained in papers in [1, 2, 6, 9, 14, 18]. The main results are Theorems 5.1 and 5.4 which give the solution to the Problem $(*)$ stated in Section 5. The technique that we developed makes references to rather many objects and we are aware of the fact that it requires some effort in order to understand the meaning of the mentioned theorems, so we start by presenting our topics in the most simple but still relevant situation.

For a selfadjoint operator $A \in \mathscr{L}(\mathscr{H})$, where $\mathscr{H}$ is a Hilbert space, we denote by $\kappa^-(A)$ the dimension of its spectral projection onto the semi-axis $(-\infty, 0)$. If $\kappa^-(A)$ is finite then it coincides with the number of negative eigenvalues, counted according to their multiplicities, of the operator A, and also it coincides with the number of negative squares of the quadratic form $\mathscr{H} \ni x \mapsto (Ax, x)$.

Let us consider Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{H}'_1$, and $\mathscr{H}'_2$.

Denote $\tilde{\mathscr{H}}_1 = \mathscr{H}_1 \oplus \mathscr{H}'_1$ and $\tilde{\mathscr{H}}_2 = \mathscr{H}_2 \oplus \mathscr{H}'_2$ and let us assume that there are given two operators $T_r \in \mathscr{L}(\tilde{\mathscr{H}}_1, \mathscr{H}_2)$ and $T_c \in \mathscr{L}(\mathscr{H}_1, \tilde{\mathscr{H}}_2)$ such that

$$T_r \mid \mathscr{H}_1 = P^{\tilde{\mathscr{H}}_2}_{\mathscr{H}_2} T_c = T \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2) \tag{1.1}$$

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and

$$\kappa^-(I-T_r^*T_r)=\kappa^-(I-T_c^*T_c)=\kappa<\infty. \tag{1.2}$$

(At this point it is worth noting that, as a consequence of [3, Proposition 3.1], for any $X\in\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)$ we have the identity $\kappa^-(I-X^*X)=\kappa^-(I-XX^*)$.) In (1.1), and throughout this paper, $P^{\tilde{\mathscr{H}}_2}_{\mathscr{H}_2}$ stands for the orthogonal projection of $\tilde{\mathscr{H}}_2$ onto $\mathscr{H}_2$. We consider the following problem of lifting of operators:

Determine all operators $\tilde{T}\in\mathscr{L}(\tilde{\mathscr{H}}_1,\tilde{\mathscr{H}}_2)$ *such that* $\tilde{T}\mid\mathscr{H}_1=T_c$, $P^{\tilde{\mathscr{H}}_2}_{\mathscr{H}_2}\tilde{T}=T_r$ *and* $\kappa^-(I-\tilde{T}^*\tilde{T})=\kappa$. $(*)'$

Here we have to mention that we use the word "determine" for two questions: first, one asks for necessary and sufficient conditions in order for there to exist at least one $\tilde{T}$ with the required properties, and second, we search for a parametrization of the set of solutions in terms of objects as simple as possible.

From (1.1) T_r and T_c can be represented as a row-matrix, respectively column-matrix,

$$T_r=[T\quad X],\qquad T_c=[T\quad Y]^t, \tag{1.3}$$

where "t" denotes the matrix transpose. Thus, Problem $(*)'$ is equivalent to determining all operators $Z\in\mathscr{L}(\mathscr{H}'_1,\mathscr{H}'_2)$ such that, denoting

$$\tilde{T}=\begin{bmatrix} T & X\\ Y & Z\end{bmatrix}, \tag{1.4}$$

the condition $\kappa^-(I-\tilde{T}^*\tilde{T})=\kappa$ holds. Define the following spaces

$$\mathscr{L}_r=P^{\mathscr{H}_2}_{\ker(I-TT^*)}X\mathscr{H}'_1\subseteq\ker(I-TT^*) \tag{1.5}$$

and

$$\mathscr{L}_c=P^{\mathscr{H}_1}_{\ker(I-T^*T)}Y^*\mathscr{H}'_2\subseteq\ker(I-T^*T). \tag{1.6}$$

Also, let us assume that the following conditions hold:

$$P_{\overline{\mathscr{R}(I-TT^*)}}\mathscr{R}(X^*)\subseteq\mathscr{R}(|I-TT^*|^{1/2}), \tag{1.7}$$

$$P_{\overline{\mathscr{R}(I-T^*T)}}\mathscr{R}(Y)\subseteq\mathscr{R}(|I-T^*T|^{1/2}). \tag{1.8}$$

The existence of solutions for Problem $(*)'$ can be settled thus:

1.1. THEOREM. *In order that Problem* $(*)'$ *has at least one solution it is necessary and sufficient that*

$$T\mathscr{L}_c=\mathscr{L}_r.$$

In order to prove this theorem, it is necessary to study first row (or column) extensions with control of the negative signatures of defect. A key step is the identification of the defect operators of a row extension in terms of the parameters. In its turn, this requires one to solve some factorization problems as in Lemmas 2.1–2.3 and also to find the range of the negative signature of some selfadjoint operatorial completions (see Lemmas 2.4 and 2.5).

Once these things are clarified, we can solve completely the problems concerning row extensions (see Section 3).

The proof of Theorem 5.1 can be done by applying row or column extensions several times. However, at the final step one encounters the problem of a row extension with data on a degenerate subspace.

The investigations of this case are done in Section 4. They make it possible to solve completely the lifting problem, i.e., to describe in terms of parameters the set of all solutions (see Theorem 5.4).

In the last Section we apply our Theorems 5.1 and 5.4 to a problem stated in [18] and derive a variant of the commutant lifting theorem of Sarason-Sz.-Nagy-Foiaş. This variant of the lifting theorem can be used in connection with some problems in [1] and [16] as is showed at the end of Section 6.

For the case $\kappa = 0$, Problem $(*)'$ is well known and it was considered, and eventually settled, in many papers [15, 21, 20, 24, 4, 12]. For the case $\kappa > 0$ (and the underlying spaces of Pontryagin or Krein type), if the condition $\kappa^-(I - T^*T) = \kappa$ holds, then this problem is solved in [3, 11] (for $\kappa = 0$ and Krein spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ in [13]). In all these cases, the problem always has solutions because both of $\mathscr{L}_r$ and $\mathscr{L}_c$ are null.

2. Preliminaries

2.1. As general references for linear operators on Krein spaces we recommend the monographs [7, 5]. Since there is no general agreement on terminology in this field we fix the notation which will be used in this paper.

A *Krein space* $\mathscr{K}$ is a complex vector space endowed with an *indefinite inner product* denoted always by $[\cdot, \cdot]$ such that there exists a decomposition $\mathscr{K} = \mathscr{K}^+ + \mathscr{K}^-$, called *fundamental decomposition*, with the properties: $\mathscr{K}^+$ and $\mathscr{K}^-$ are linear manifolds of $\mathscr{K}$, orthogonal with respect to $[\cdot, \cdot]$ and $(\mathscr{K}^+, [\cdot, \cdot])$, $(\mathscr{K}^-, -[\cdot, \cdot])$ are Hilbert spaces. We use the notation $\mathscr{K} = \mathscr{K}^+ [+] \mathscr{K}^-$ to denote a fundamental decomposition. With respect to any fundamental decomposition $\mathscr{K} = \mathscr{K}^+ [+] \mathscr{K}^-$ one associates a *fundamental symmetry* (f.s. for short) J on $\mathscr{K}$, defined by

$$J(x^+ + x^-) = x^+ - x^-, \qquad x^\pm \in \mathscr{K}^\pm.$$

The corresponding *J-inner product* $(\cdot, \cdot)_J$, defined by

$$(x, y)_J = [Jx, y]$$

turns $\mathcal{K}$ into a Hilbert space. The norm $\|\cdot\|$ associated with the J-inner product is called a *unitary norm* of $\mathcal{K}$. Any two unitary norms on a Krein space $\mathcal{K}$ are equivalent, in particular the *strong topology* on $\mathcal{K}$ is determined by an arbitrary unitary norm. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are Krein spaces then $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ denotes the set of linear bounded operators acting from $\mathcal{K}_1$ into $\mathcal{K}_2$.

The *signature numbers* $\kappa^{\pm}[\mathcal{K}] = \dim \mathcal{K}^{\pm}$, where $\mathcal{K} = \mathcal{K}^{+}[+]\mathcal{K}^{-}$ is an arbitrary fundamental decomposition of $\mathcal{K}$ and the symbol dim denotes the Hilbert space dimension, are independent on the chosen fundamental decomposition. If the rank of indefiniteness $\kappa(\mathcal{K}) = \min\{\kappa^{+}[\mathcal{K}], \kappa^{-}[\mathcal{K}]\}$ is finite then $\mathcal{K}$ is called a Pontryagin space (in most of the cases this will mean $\kappa^{-}[\mathcal{K}]$ finite).

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two Krein spaces and consider J_1 and J_2 two f.s.'s on $\mathcal{K}_1$ and respectively $\mathcal{K}_2$. With respect to the Hilbert space $(\mathcal{K}_1, (\cdot, \cdot)_{J_1})$ and $(\mathcal{K}_2, (\cdot, \cdot)_{J_2})$ we consider the direct sum Hilbert space $\mathcal{K}_1 \oplus \mathcal{K}_2$. Then $J = J_1 \oplus J_2$ is a symmetry on this Hilbert space (recall that $J \in \mathcal{L}(\mathcal{H})$, for $\mathcal{H}$ a Hilbert space, is a *symmetry* if $J = J^* = J^{-1}$) and defining the inner product $[\cdot, \cdot]$ on $\mathcal{K}_1 \oplus \mathcal{K}_2$ by

$$[x, y] = (Jx, y), \qquad x, y \in \mathcal{K}_1 \oplus \mathcal{K}_2,$$

$(\mathcal{K}_1 \oplus \mathcal{K}_2, [\cdot, \cdot])$ becomes a Krein space and $J_1 \oplus J_2$ is a f.s. of it. This construction does not depend on the f.s.'s J_1 and J_2, and the *direct sum Krein space* obtained in this way will be denoted by $\mathcal{K}_1[+]\mathcal{K}_2$.

For any operator $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$, $\mathcal{K}_1$ and $\mathcal{K}_2$ being Krein spaces, one defines its *adjoint* $T^{\#} \in \mathcal{L}(\mathcal{K}_2, \mathcal{K}_1)$ by

$$[Tx, y] = [x, T^{\#} y], \qquad x \in \mathcal{K}_1,\ y \in \mathcal{K}_2.$$

A possibly unbounded operator T with domain $\mathcal{D}(T) \subseteq \mathcal{K}_1$ and range $\mathcal{R}(T) \subseteq \mathcal{K}_2$ is called *isometric* if

$$[Tx, Ty] = [x, y], \qquad x, y \in \mathcal{D}(T).$$

A surjective isometry $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ is called *unitary.* In order to exist unitary operators acting between $\mathcal{K}_1$ and $\mathcal{K}_2$ it is necessary and sufficient that $\kappa^{\pm}[\mathcal{K}_1] = \kappa^{\pm}[\mathcal{K}_2]$.

An operator $A \in \mathcal{L}(\mathcal{K})$, $\mathcal{K}$ being a Krein space, is called *selfadjoint* if $A = A^{\#}$ and it is called *positive* if

$$[Ax, x] \geqslant 0, \qquad x \in \mathcal{K}.$$

An operator $T \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$ is *contractive* if

$$[Tx, Tx] \leqslant [x, x], \qquad x \in \mathscr{K}_1,$$

equivalently $I - T^{\#} T$ is positive. T is called *doubly contractive* if both of T and $T^{\#}$ are contractive.

If J_1 and J_2 are f.s.'s on $\mathscr{K}_1$ and respectively $\mathscr{K}_2$, let T^* denote the adjoint of $T \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$ with respect to the Hilbert spaces $(\mathscr{K}_i, (\cdot, \cdot)_{J_i})$, $i = 1, 2$. Then we have $T^{\#} = J_1 T^* J_2$ and thus most of the above definitions can be translated into the language of linear operators on Hilbert spaces. In order to avoid complications with transfinite cardinals, all the Krein or Hilbert spaces appearing in this paper will be assumed to be *separable* (though most of the results are true also without this assumption).

2.2. Let $A \in \mathscr{L}(\mathscr{H})$ be a selfadjoint operator, $\mathscr{H}$ being a Hilbert space. Denote by sgn the function signum (by definition, this function vanishes at 0). Denote $S_A = \operatorname{sgn}(A)$; then S_A is the selfadjoint partial isometry appearing in the polar decomposition of A,

$$A = S_A |A|, \qquad \ker S_A = \ker A, \qquad S_A \mathscr{H} = \overline{\mathscr{R}(A)}.$$

The signature numbers of A are defined as follows:

$$\kappa^{\pm}(A) = \operatorname{dimker}(I \mp S_A), \qquad \kappa^0(A) = \operatorname{dimker} S_A. \tag{2.1}$$

Denote by $\mathscr{H}_A$ the Krein space obtained from $S_A \mathscr{H}$ endowed with the inner product $[\cdot, \cdot]$,

$$[x, y] = (S_A x, y), \qquad x, y \in \mathscr{H}$$

(here $(\cdot, \cdot)$ denotes the inner product of the Hilbert space $\mathscr{H}$).

With this definition it is clear that $\kappa^{\pm}(A) = \kappa^{\pm}[\mathscr{H}_A]$.

In this paper we will frequently use two results concerning indefinite factorizations. The first one is well known [22, 8] (see also [3]).

2.1. Lemma. *Let $A \in \mathscr{L}(\mathscr{H})$ be selfadjoint, $\mathscr{H}$ a Hilbert space and $\mathscr{K}$ a Krein space with J a f.s. fixed on it. Then there exists $B \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ such that*

$$A = B^* J B \tag{2.2}$$

if and only if

$$\kappa^{\pm}[\mathscr{K}] \geqslant \kappa^{\pm}(A). \tag{2.3}$$

The next result is taken from [11, Corollary 1.3].

2.2. LEMMA. *Let A, $\mathcal{H}$ and $\mathcal{K}$ be as in Lemma* 2.1. *Assume that either $\kappa^-(A)$ or $\kappa^+(A)$ is finite and there exists $B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ with dense range and satisfying* (2.2). *Then B is of the form $B = U|A|^{1/2}$, where $U \in \mathcal{L}(\mathcal{H}_A, \mathcal{K})$ is a uniquely determined unitary operator (of Pontryagin spaces).*

In this paper we also need the following result.

2.3. LEMMA. *Let $A \in \mathcal{L}(\mathcal{H})$ be selfadjoint, $\mathcal{H}$ and $\mathcal{K}$ Hilbert spaces. If $B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ satisfies*

$$P_{\mathcal{H}_A}\mathcal{R}(B^*) \subseteq \mathcal{R}(|A|^{1/2}) \tag{2.4}$$

then, with respect to the decomposition $\mathcal{H} = \mathcal{H}_A \oplus \ker A$, B has the representation

$$B = [X|A|^{1/2} \quad Y], \tag{2.5}$$

where $X \in \mathcal{L}(\mathcal{H}_A, \mathcal{K})$ and $Y \in \mathcal{L}(\ker A, \mathcal{K})$ are uniquely determined.

If, in addition, $\kappa^-(A - B^*B) < \infty$ then Y has finite rank and $\kappa^-(S_A - X^*X) < \infty$.

Proof. Define $Y = B|\ker A \in \mathcal{L}(\ker A, \mathcal{K})$ and $X: \mathcal{R}(|A|^{1/2}) \to \mathcal{K}$ by

$$X|A|^{1/2}h = Bh, \qquad h \in \mathcal{H}_A. \tag{2.6}$$

Since $|A|^{1/2}$ is injective on $\mathcal{H}_A = \mathcal{H} \ominus \ker A$, the linear mapping X is correctly defined by (2.6) and (2.5) holds. It remains to notice that by (2.4) X is bounded on $\mathcal{R}(|A|^{1/2})$ and then we will extend it by continuity to a uniquely determined operator, also denoted by X, in $\mathcal{L}(\mathcal{H}_A, \mathcal{K})$.

Finally, considering the representation

$$A - B^*B = \begin{bmatrix} |A|^{1/2}(S_A - X^*X)|A|^{1/2} & -|A|^{1/2}X^*Y \\ -Y^*X|A|^{1/2} & -Y^*Y \end{bmatrix}$$

with respect to the decomposition $\mathcal{H} = \mathcal{H}_A \oplus \ker A$, application of Lemmas 2.1 and 2.2 shows that Y has finite rank and $\kappa^-(S_A - X^*X) < \infty$, provided that $\kappa^-(A - B^*B) < \infty$. ∎

2.3. Two results concerning the range of the negative signatures of some operator-matrix completions will be frequently used in the paper. The first one is a simple application of the *Schur reduction* (performing a Schur reduction will always mean to use one of the following dual identities: for

A, B, C, and D bounded operators on appropriate Hilbert spaces, if A is invertible then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} \tag{2.7}$$

and if D is invertible,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}, \tag{2.8}$$

where $D - CA^{-1}B$ and $A - BD^{-1}C$ are usually called *Schur complements*).

2.4. LEMMA. *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two Hilbert spaces and $C \in \mathscr{L}(\mathscr{G}_2)$ a selfadjoint operator. Then*

$$\left\{ \kappa^- \left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \right) \middle| A = A^* \in \mathscr{L}(\mathscr{G}_1) \text{ and } B \in \mathscr{L}(\mathscr{G}_2, \mathscr{G}_1) \right\}$$
$$= \{ \kappa \in \tilde{\mathbb{N}} \ (= \mathbb{N} \cup \{\infty\}) \mid \kappa^-(C) \leqslant \kappa \leqslant \kappa^-(C) + \dim \mathscr{G}_1 \}.$$

Proof. The inequality

$$\kappa^-(C) \leqslant \kappa^- \left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \right)$$

is obvious. The other one is obvious when $\dim \mathscr{G}_1$ or $\kappa^-(C)$ are infinite. So let us suppose that both of them are finite. For fixed A and B, we can find $\varepsilon > 0$ sufficiently small such that $C + \varepsilon$ is invertible, $\kappa^-(C) = \kappa^-(C + \varepsilon)$ and

$$\kappa^- \left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \right) = \kappa^- \left(\begin{bmatrix} A + \varepsilon & B \\ B^* & C + \varepsilon \end{bmatrix} \right).$$

Performing a Schur reduction, we obtain that

$$\kappa^- \left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \right) = \kappa^- \left(\begin{bmatrix} A + \varepsilon - B(C+\varepsilon)^{-1}B^* & 0 \\ 0 & C + \varepsilon \end{bmatrix} \right) \leqslant \dim \mathscr{G}_1 + \kappa^-(C).$$

In this way, we obtained one of the inclusions in the statement of the lemma. For the other one, observe first that for $\kappa^-(C)$ infinite this is obvious, so let us assume $\kappa^-(C) < \infty$.

In this case, take $\kappa \in \tilde{\mathbb{N}}$ such that

$$\kappa^-(C) \leqslant \kappa \leqslant \kappa^-(C) + \dim \mathscr{G}_1$$

or, what is the same,

$$0 \leqslant \kappa - \kappa^-(C) \leqslant \dim \mathscr{G}_1.$$

Consequently, there exists $X \in \mathscr{L}(\mathscr{G}_1)$, $X = X^*$ such that $\kappa^-(X) = \kappa - \kappa^-(C)$ and

$$\kappa^-\left(\begin{bmatrix} X & 0 \\ 0 & C \end{bmatrix}\right) = \kappa^-(X) + \kappa^-(C) = \kappa. \quad \blacksquare$$

The second result is of different nature.

2.5. LEMMA. *Let $\mathscr{H}$ and $\mathscr{G}$ be Hilbert spaces, S a selfadjoint operator in $\mathscr{L}(\mathscr{H})$ and J a symmetry in $\mathscr{L}(\mathscr{G})$. Then*

$$\begin{aligned} &\{\kappa^-(S - U^*JU) \mid U \in \mathscr{L}(\mathscr{H}, \mathscr{G})\} \\ &\quad = \{\kappa \in \tilde{\mathbb{N}} \mid \max\{0, \kappa^-(S) - \kappa^-(J)\} \leqslant \kappa \\ &\quad \leqslant \kappa^-(S) + \min\{\kappa^+(J), \kappa^+(S) + \kappa^0(S)\}\} \end{aligned}$$

with the convention that $\kappa^-(S) - \kappa^-(J) = 0$, when both of them are infinite.

Proof. For $\kappa^-(S) < \kappa^-(J)$, we obviously have $\kappa^-(S - U^*JU) \geqslant 0$, so, let us suppose $\kappa^-(S) \geqslant \kappa^-(J)$. Let $S = S^+ - S^-$ and $J = J^+ - J^-$ be the Jordan decompositions and $U \in \mathscr{L}(\mathscr{H}, \mathscr{G})$. Then

$$S - U^*JU = S^+ - S^- - U^*J^+U + U^*J^-U \tag{2.9}$$

and, as $S^+S^- = 0$, it results that

$$\kappa^-(S - U^*JU) \geqslant \kappa^-(S) - \kappa^-(J).$$

Anyway,

$$\kappa^-(S - U^*JU) \geqslant \max\{0, \kappa^-(S) - \kappa^-(J)\}.$$

Also by (2.9),

$$\begin{aligned} \kappa^-(S - U^*JU) &\leqslant \kappa^-(S) + \min\{\operatorname{rank}(U^*J^+U), \kappa^+(S) + \kappa^0(S)\} \\ &\leqslant \kappa^-(S) + \min\{\kappa^+(J), \kappa^+(S) + \kappa^0(S)\}, \end{aligned}$$

and one inclusion in the statement of the lemma is proved. For the other one, take $\kappa \in \tilde{\mathbb{N}}$ such that

$$\max\{0, \kappa^-(S) - \kappa^-(J)\} \leqslant \kappa \leqslant \kappa^-(S) + \min\{\kappa^+(J), \kappa^+(S) + \kappa^0(S)\}.$$

We distinguish two possibilities: $\kappa \geqslant \kappa^-(S)$ and $\kappa \leqslant \kappa^-(S)$. In the first case, when $\kappa^-(S)$ is infinite, we can choose $U=0$; when $\kappa^-(S)<\infty$, then

$$0 \leqslant \kappa - \kappa^-(S) \leqslant \min\{\kappa^+(J), \kappa^+(S)+\kappa^0(S)\}$$

and there exists a spectral projection Q of S with rank $Q=\kappa-\kappa^-(S)$ and $QS^-=0$. By Lemma 2.1, there exists $U \in \mathscr{L}(\mathscr{H}, \mathscr{G})$ such that $U^*J^-U=0$ and $U^*J^+U=(\|S\|+1)Q$. Consequently,

$$\begin{aligned}\kappa^-(S-U^*JU) &= \kappa^-((S^+-(\|S\|+1)Q)-S^-) = \operatorname{rank} Q+\kappa^-(S)\\ &= \kappa-\kappa^-(S)+\kappa^-(S)=\kappa.\end{aligned}$$

In the second case, $\kappa \leqslant \kappa^-(S)$, there exists a spectral projection R of S with rank $R=\operatorname{rank}(S^-R)=\kappa$. Again by Lemma 2.1, there exists $U \in \mathscr{L}(\mathscr{H}, \mathscr{G})$ such that $U^*JU=-(I-R)S^-$. Consequently,

$$\kappa^-(S-U^*JU)=\kappa^-(-RS^-)=\kappa. \quad \blacksquare$$

2.4. Let $\mathscr{K}$ be a Krein space and $A \in \mathscr{L}(\mathscr{K})$, $A=A^\#$. Then, for an arbitrary f.s. J on $\mathscr{K}$ the operator $JA \in \mathscr{L}(\mathscr{K})$ is selfadjoint with respect to the inner product $(\cdot,\cdot)_J$ and

$$(JAx, y)_J=[Ax, y], \qquad x, y \in \mathscr{K}.$$

In particular, recalling the definitions from (2.1), the signature numbers $\kappa^\pm[A]$ and $\kappa^0[A]$ defined by

$$\kappa^\pm[A]=\kappa^\pm(JA), \qquad \kappa^0[A]=\kappa^0(JA) \tag{2.10}$$

are independent of the chosen f.s. J.

Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be Krein spaces and fix J_1 and J_2 two f.s.'s on $\mathscr{K}_1$ and respectively $\mathscr{K}_2$. For an arbitrary operator $T \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$, the *signature numbers of defect* are

$$\kappa^\pm[I-T^\# T]=\kappa^\pm(J_1-T^*J_2T) \tag{2.11}$$

$$\kappa^0[I-T^\# T]=\kappa^0(J_1-T^*J_2T). \tag{2.12}$$

From [3, Proposition 3.1] the following identities hold

$$\kappa^\pm[I-T^\# T]+\kappa^\pm[\mathscr{K}_2]=\kappa^\pm[I-TT^\#]+\kappa^\pm[\mathscr{K}_1] \tag{2.13}$$

$$\kappa^0[I-T^\# T]=\kappa^0[I-TT^\#]. \tag{2.14}$$

Also, let us introduce the operator J_T by

$$J_T=\operatorname{sgn}(J_1-T^*J_2T) \tag{2.15}$$

and the so-called *defect operator* D_T by

$$D_T = |J_1 - T^*J_2T|^{1/2}. \tag{2.16}$$

Then the *defect subspace* $\mathscr{D}_T$ is introduced as the Krein space $\overline{\mathscr{R}(D_T)}$ with the f.s. J_T. According to [3, Proposition 4.1] there exist (the so-called *link operators*) $L_T \in \mathscr{L}(\mathscr{D}_T, \mathscr{D}_{T^*})$ and $L_{T^*} \in \mathscr{L}(\mathscr{D}_{T^*}, \mathscr{D}_T)$, uniquely determined such that

$$D_{T^*}L_T = TJ_1D_T | \mathscr{D}_T, \; D_TL_{T^*} = T^*J_2D_{T^*} | \mathscr{D}_{T^*}. \tag{2.17}$$

Then, also by [3], the following identities hold

$$L_{T^*} = J_TL_T^*J_{T^*} | \mathscr{D}_{T^*}, \tag{2.18}$$

$$(J_T - D_TJ_1D_T) | \mathscr{D}_T = L_T^*J_{T^*}L_T. \tag{2.19}$$

From (2.17)–(2.19) it follows that the operator $R(T) \in \mathscr{L}(\mathscr{K}_1[+]\mathscr{D}_{T^*}, \mathscr{K}_2[+]\mathscr{D}_T)$ defined by

$$R(T) = \begin{bmatrix} T & D_{T^*} \\ D_T & -J_TL_{T^*} \end{bmatrix} \tag{2.20}$$

is a unitary operator of Krein spaces and it is called the *elementary rotation* of T.

2.6. *Remark.* From the obvious identity

$$TJ_1(J_1 - T^*J_2T) = (J_2 - TJ_1T^*)J_2T,$$

we infer

$$(J_2T) \ker D_T = \ker D_{T^*}, \qquad (J_1T^*) \ker D_{T^*} = \ker D_T,$$

and the operator $(J_1T^*) | \ker D_{T^*} \in \mathscr{L}(\ker D_{T^*}, \ker D_T)$ is the inverse of the operator $(J_2T) | \ker D_T \in \mathscr{L}(\ker D_T, \ker D_{T^*})$.

Throughout this paper, whenever a Krein space will be considered, a f.s. will be fixed on it and with respect to this will be considered defect operators, defect subspaces, etc. The Krein space $\mathscr{D}_T$ will be always considered with the fixed f.s. J_T.

3. Row and Column Extensions

In this section we consider the following problem of extending operators with prescribed negative signatures of defect.

Let $\mathscr{K}_1$. $\mathscr{K}_2$, and $\mathscr{K}'_1$ be Krein spaces and consider the Krein space

$\tilde{\mathscr{H}}_1 = \mathscr{H}_1[+]\mathscr{H}'_1$. Let also $T \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ and $\kappa_1, \kappa_2 \in \tilde{\mathbb{N}}$ be given. The problem has the statement:

Determine all the operators $T_r \in \mathscr{L}(\tilde{\mathscr{H}}_1, \mathscr{H}_2)$ *with* $T_r | \mathscr{H}_1 = T$, $\kappa^-[I - T_r^{\#} T_r] = \kappa_1$, *and* $\kappa^-[I - T_r T_r^{\#}] = \kappa_2$. $\quad (*)_r$

We solve this problem for the case when $\mathscr{H}'_1$ is a Hilbert space and κ_2 is finite. Let us also fix some more notation.

If $B \in \mathscr{L}(\mathscr{G}_1, \mathscr{G}_2)$ has closed range, $\mathscr{G}_1$ and $\mathscr{G}_2$ being Hilbert spaces, by B^{-1} we always denote the uniquely determined operator acting from $\mathscr{R}(B)$ and valued in $\mathscr{G}_1 \ominus \ker B$, the inverse of $B | \mathscr{G}_1 \ominus \ker B$.

Let $\mathscr{G}$ be a Hilbert space. Then $[-\mathscr{G}]$ will denote the Krein space which is the anti-space of $\mathscr{G}$, i.e., $-I$ is the f.s. Also, by $[\mathscr{G} \oplus \mathscr{G}]$ we denote the Krein space whose indefinite inner product is determined by the symmetry

$$J_0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{3.1}$$

From now on we fix two f.s.'s J_1 and J_2 on $\mathscr{H}_1$ and respectively $\mathscr{H}_2$. Then $\tilde{J}_1 = J_1 \oplus I$ (recall that $\mathscr{H}'_1$ was assumed a Hilbert space) is the f.s. which will be fixed on the Krein space $\tilde{\mathscr{H}}_1 = \mathscr{H}_1[+]\mathscr{H}'_1$. With respect to these f.s.'s will be considered defect operators, defect spaces, etc.

3.1. Lemma. *Let* $T_r \in \mathscr{L}(\tilde{\mathscr{H}}_1, \mathscr{H}_2)$ *be an extension of* T, *i.e.*, $T_r | \mathscr{H}_1 = T$. *If we suppose*

$$P_{\overline{\mathscr{R}(D_{T^*})}} \mathscr{R}(T_r^*) \subseteq \mathscr{R}(D_{T^*}) \tag{3.2}$$

then there exist uniquely determined $\Lambda \in \mathscr{L}(\mathscr{H}'_1, \ker D_{T^*})$, $\Gamma \in \mathscr{L}(\ker \Lambda, \mathscr{D}_{T^*})$, *and* $\Delta \in \mathscr{L}(\mathscr{R}(\Lambda^*), \mathscr{D}_{T^*})$, *such that with respect to the decompositions*

$$\mathscr{H}'_1 = \ker \Lambda \oplus \overline{\mathscr{R}(\Lambda^*)}, \qquad \mathscr{H}_2 = \mathscr{D}_{T^*} \oplus \ker D_{T^*} \tag{3.3}$$

we have

$$T_r | \mathscr{H}'_1 = \begin{bmatrix} D_{T^*}\Gamma & D_{T^*}\Delta \\ 0 & \Lambda \end{bmatrix}. \tag{3.4}$$

Moreover, if $\kappa^-[I - T_r T_r^{\#}] < \infty$ *then there exists a unique unitary operator* $U_*(T_r)$ (*between Pontryagin spaces*) *such that*

$$\begin{aligned} &U_*(T_r) \colon \mathscr{D}_{T_r^*} \to \mathscr{D}_{\Gamma^*}[+][-\mathscr{R}(\Lambda^*)] \\ &U_*(T_r) D_{T_r^*} = \begin{bmatrix} D_{\Gamma^*} D_{T^*} & 0 \\ \Delta^* D_{T^*} & \Lambda^* \end{bmatrix} = D_{*,r} \end{aligned} \tag{3.5}$$

and the following identity holds

$$\kappa^-[I - T_r^\# T_r] = \kappa^-[I - T^\# T] + \kappa^-[I - \Gamma^\# \Gamma] + \operatorname{rank}(\Lambda). \tag{3.6}$$

In particular, $\kappa^-[I - T_r^\# T_r]$ *and* $\kappa^-[I - T^\# T]$ *are simultaneously finite and in this case there exists a unique unitary operator* $U(T_r)$ *(between Pontryagin spaces) such that*

$$U(T_r)\colon \mathscr{D}_{T_r} \to \mathscr{D}_T[+][\mathscr{R}(\Lambda) \oplus \mathscr{R}(\Lambda)][+]\mathscr{D}_\Gamma$$

$$U(T_r) D_{T_r} = \begin{bmatrix} D_T & 0 & -J_T L_{T^*} \Delta & -J_T L_{T^*} \Gamma \\ 0 & -P^{\mathscr{H}_2}_{\mathscr{R}(\Lambda)} J_2 T & \frac{1}{2}\Lambda^{*-1} E & -\Lambda^{*-1} \Delta^* J_{T^*} \Gamma \\ 0 & 0 & \Lambda & 0 \\ 0 & 0 & 0 & D_\Gamma \end{bmatrix} = D_r, \tag{3.7}$$

where

$$E = I - \Lambda^* P^{\mathscr{H}_2}_{\mathscr{R}(\Lambda)} J_2 \Lambda - \Delta^* J_{T^*} \Delta.$$

Proof. Let $T_r \in \mathscr{L}(\tilde{\mathscr{H}}_1, \mathscr{H}_2)$ satisfy $T_r \mid \mathscr{H}_1 = T$. Then, with respect to the decomposition $\tilde{\mathscr{H}}_1 = \mathscr{H}_1[+]\mathscr{H}'_1$, it is represented as

$$T_r = [T \quad S]. \tag{3.8}$$

Then

$$J_2 - T_r \tilde{J}_1 T_r^* = J_2 - T J_1 T^* - S S^*, \tag{3.9}$$

hence, if (3.2) holds, application of Lemma 2.3 yields the representation (3.4), where

$$\begin{gathered} \Lambda^* = P^{\mathscr{H}_1}_{\mathscr{H}'_1} T_r^* \mid \ker D_{T^*}, \qquad \Gamma^* D_{T^*} = P^{\mathscr{H}_1}_{\ker \Lambda} T_r^* \mid \mathscr{D}_{T^*}, \\ \Delta^* D_{T^*} = P^{\mathscr{H}_1}_{\mathscr{R}(\Lambda^*)} T_r^* \mid \mathscr{D}_{T^*} \end{gathered} \tag{3.10}$$

with the remark that if $\kappa^-[I - T_r T_r^\#] < \infty$ then Λ^* has finite rank and thus its range is closed. Inserting (3.4) in (3.9) we obtain

$$J_2 - T_r \tilde{J}_1 T_r^* = \begin{bmatrix} J_2 - T J_1 T^* - D_{T^*}(\Gamma\Gamma^* + \Delta\Delta^*) D_{T^*} & -D_{T^*} \Delta \Lambda^* \\ -\Lambda \Delta^* D_{T^*} & -\Lambda\Lambda^* \end{bmatrix},$$

where the block-matrix is written with respect to the decomposition $\mathscr{H}_2 = \mathscr{D}_{T^*} \oplus \ker D_{T^*}$. From here it is easy to obtain the factorization

$$J_2 - T_r \tilde{J}_1 T_r^* = D_{*,r}^* \begin{bmatrix} J_{\Gamma^*} & 0 \\ 0 & -I \end{bmatrix} D_{*,r}, \tag{3.11}$$

where $D_{*,r} \in \mathscr{L}(\mathscr{D}_{T^*} \oplus \ker D_{T^*}, \mathscr{D}_{\Gamma^*} \oplus \mathscr{R}(\Lambda^*))$ was defined by (3.5).

Since $D_{*,r}$ has dense range, application of Lemma 2.2 shows that there exists a uniquely determined unitary operator of Pontryagin spaces, $U_*(T_r)$, satisfying (3.5).

Further on, considering the operator D_r defined at (3.7), we will prove that the following factorization holds,

$$\tilde{J}_1 - T_r^* J_2 T_r = D_r^* (J_T \oplus J_0 \oplus J_\Gamma) D_r, \tag{3.12}$$

where J_0 is defined by (3.1) with respect to the space $\mathscr{R}(\Lambda)$.

To this end, let us remark that taking account of the representation (3.4) of S, a direct computation shows that $\tilde{J}_1 - T_r^* J_2 T_r$ has the block-matrix

$$\left[\begin{array}{cc|cc} J_1 - T^* J_2 T & 0 & -T^* J_2 D_{T^*} \Delta & -T^* J_2 D_{T^*} \Gamma \\ 0 & 0 & -T^* J_2 \Lambda & 0 \\ \hline -\Delta^* D_{T^*} J_2 T & -\Lambda^* J_2 T & I - \Delta^* D_{T^*} J_2 D_{T^*} \Delta - \Lambda^* J_2 \Lambda & -\Delta^* D_{T^*} J_2 D_{T^*} \Gamma \\ -\Gamma^* D_{T^*} J_2 T & 0 & -\Gamma^* D_{T^*} J_2 D_{T^*} \Delta & I - \Gamma^* D_{T^*} J_2 D_{T^*} \Gamma \end{array}\right]$$

with respect to the decomposition

$$\tilde{\mathscr{H}}_1 = \mathscr{D}_T \oplus \ker D_T \oplus \mathscr{R}(\Lambda^*) \oplus \ker \Lambda.$$

Now, from the definition of D_r we obtain

$$D_r^* [J_T \oplus J_0 \oplus J_\Gamma] D_r = \left[\begin{array}{c|c} A & B \\ \hline B^* & C \end{array}\right], \tag{3.13}$$

where

$$A = \begin{bmatrix} D_T J_T D_T & 0 \\ 0 & 0 \end{bmatrix} \tag{3.14}$$

$$B = \begin{bmatrix} -D_T L_{T^*} \Delta & -D_T L_{T^*} \Gamma \\ -T^* J_2 \Lambda & 0 \end{bmatrix} \tag{3.15}$$

$$C = \begin{bmatrix} I + \Delta^* L_{T^*}^* J_T L_{T^*} \Delta - \Delta^* J_{T^*} \Delta - \Lambda^* J_2 \Lambda & \Delta^* L_{T^*}^* J_T L_{T^*} \Gamma - \Delta^* J_{T^*} \Gamma \\ \Gamma^* L_{T^*}^* J_T L_{T^*} \Delta - \Gamma^* J_{T^*} \Delta & \Gamma^* L_{T^*}^* J_T L_{T^*} \Gamma + D_\Gamma J_\Gamma D_\Gamma \end{bmatrix}. \tag{3.16}$$

Using the properties (2.17)–(2.19) of the link operator L_{T^*}, the correctness of the factorization (3.12) follows from the structure of $\tilde{J}_1 - T_r^* J_2 T_r$ and (3.13)–(3.16).

Let us remark that D_r has dense range (using Remark 2.6).

Then an argument of Pontryagin Lemma type (see [7, Theorem IX.1.4]) as in the proof of [11, Corollary 1.3], applied to the factorization (3.12), shows that the identity (3.6) is true.

Taking account of the obvious inequality

$$\kappa^-[I - T^\# T] \leqslant \kappa^-[I - T_r^\# T_r] \tag{3.17}$$

if either $\kappa^-[I - T^\# T]$ or $\kappa^-[I - T_r^\# T_r]$ is finite then the application of Lemma 2.2 to the factorization (3.12) proves that (3.7) uniquely determines a unitary operator $U(T_r)$ of Pontryagin spaces, in particular both of $\kappa^-[I - T^\# T]$ and $\kappa^-[I - T_r^\# T_r]$ are finite. ∎

3.2. Corollary. *Let $T_r \in \mathscr{L}(\tilde{\mathscr{H}}_1, \mathscr{H}_2)$ be such that $T_r \mid \mathscr{H}_1 = T$ and* (3.2) *holds. Then*

$$\kappa^-[I - T_r^\# T_r] + \kappa^-[I - TT^\#] = \kappa^-[I - T_r T_r^\#] + \kappa^-[I - T^\# T]. \tag{3.18}$$

Proof. Let $T_r \in \mathscr{L}(\tilde{\mathscr{H}}_1, \mathscr{H}_2)$ be an extension of T. If $\kappa^-[I - T_r T_r^\#]$ is finite, then we can use Lemma 3.1 in order to obtain the representation (3.4) and from the existence of the unitary operator $U_*(T_r)$ as in (3.5) we obtain

$$\kappa^-[I - T_r T_r^\#] = \kappa^-[I - \Gamma\Gamma\#] + \operatorname{rank}(\Lambda). \tag{3.19}$$

The identity (2.13) corresponding to Γ is

$$\kappa^-[I - \Gamma^\# \Gamma] + \kappa^-[I - TT^\#] = \kappa^-[I - \Gamma\Gamma^\#] \tag{3.20}$$

and from here, (3.19), and (3.6) we obtain (3.18) (remark, e.g., by (3.17), that $\kappa^-[I - TT^\#]$ is also finite in this case).

If $\kappa^-[I - TT^\#] = \infty$ then also $\kappa^-[I - T_r T_r^\#] = \infty$ and (3.18) is obviously true.

Finally, assume $\kappa^-[I - TT^\#] < \infty$ and $\kappa^-[I - T_r T_r^\#] = \infty$.

We prove that in this situation it is necessary that $\kappa^-[I - T_r^\# T_r] = \infty$, which clearly implies (3.18). To this end, take $\kappa \geqslant \kappa^-[I - TT^\#]$ arbitrarily large but finite. Then, there exists a finite dimensional subspace $\mathscr{L}$ of $\mathscr{H}_2$ such that, with respect to the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle h, k \rangle = ((J_2 - T_r \tilde{J}_1 T_r^*) h, k), \qquad h, k \in \mathscr{L},$$

its negative signature is at least κ. Consider $S = T_r | \mathscr{H}'_1$ and denote by P the orthogonal projection of $\mathscr{H}'_1$ onto $S^*\mathscr{L}$. Define $T_{r,P} \in \mathscr{L}(\tilde{\mathscr{H}}_1, \mathscr{H}_2)$ an extension of T by

$$T_{r,P} = [T \quad SP]$$

and remark that

$$\kappa \leqslant \kappa^-[I - T_{r,P} T^\#_{r,P}] < \infty$$

and since (3.18) was already checked for this case, we have

$$\begin{aligned}\kappa^-[I - T^\#_{r,P} T_{r,P}] &= \kappa^-[I - T_{r,P} T^\#_{r,P}] + \kappa^-[I - T^\# T] - \kappa^-[I - TT^\#] \\ &\geqslant \kappa + \kappa^-[I - T^\# T] - \kappa^-[I - TT^\#]. \qquad (3.21)\end{aligned}$$

An application of Lemma 2.1 to the factorization

$$\tilde{J}_1 - T^*_{r,P} J_2 T_{r,P} = [I \oplus P](\tilde{J}_1 - T_r^* J_2 T_r)[I \oplus P]$$

together with (3.21) imply

$$\kappa + \kappa^-[I - T^\# T] - \kappa^-[I - TT^\#] \leqslant \kappa^-[I - T_r^\# T_r]$$

and this proves that $\kappa^-[I - T_r^\# T_r] = \infty$ because κ can be chosen arbitrarily large. The identity (3.18) is proved in all possible cases. ∎

3.3. *Remark.* If $\mathscr{H}_1$ and $\mathscr{H}_2$ are Pontryagin spaces (by this we mean $\kappa^-[\mathscr{H}_1]$, $\kappa^-[\mathscr{H}_2] < \infty$) then (3.18) is a simple consequence of (2.13).

We can state now the solution of Problem $(*)_r$ in the above mentioned case.

3.4. THEOREM. *Assume $\mathscr{H}'_1$ is a Hilbert space.*
Then Problem $(*)_r$ has solutions if the conditions hold

$$\kappa_1 + \kappa^-[I - TT^\#] = \kappa_2 + \kappa^-[I - T^\# T], \qquad (3.22)$$

$$\kappa^-[I - TT^\#] \leqslant \kappa_2 \leqslant \kappa^-[I - TT^\#] + \kappa_{+0}, \qquad (3.23)$$

$$\kappa^-[I - T^\# T] \leqslant \kappa_1 \leqslant \kappa^-[I - T^\# T] + \kappa_{+0}, \qquad (3.24)$$

where

$$\kappa_{+0} = \min\{\dim \mathscr{H}'_1, \kappa^+[I - TT^\#] + \kappa^0[I - TT^\#]\}.$$

Moreover, if conditions (3.22)–(3.24) *hold, then the formula* (3.4) *establishes a bijective correspondence between a subset of solutions T_r of Problem $(*)_r$*

and the set of all triplets $\{\Lambda, \Gamma, \Delta\}$, *where* $\Lambda \in \mathscr{L}(\mathscr{H}'_1, \ker D_T)$ *and* $\Gamma \in \mathscr{L}(\ker \Lambda, \mathscr{D}_{T^*})$ *satisfy*

$$\kappa_2 = \kappa^-[I - \Gamma\Gamma^\#] + \operatorname{rank}(\Lambda) \tag{3.25}$$

and $\Delta \in \mathscr{L}(\mathscr{R}(\Lambda^*), \mathscr{D}_{T^*})$ *is arbitrary.*

Proof. Suppose (3.22)–(3.24) are fulfilled. First we prove that there exist $\Lambda \in \mathscr{L}(\mathscr{H}'_1, \ker D_T)$ and $\Gamma \in \mathscr{L}(\ker \Lambda, \mathscr{D}_{T^*})$ satisfying (3.25) and

$$\kappa_1 = \kappa^-[I - T^\# T] + \kappa^-[I - \Gamma^\# \Gamma] + \operatorname{rank}(\Lambda). \tag{3.26}$$

To this end, we distinguish two cases:

(a) $\kappa^-[I - TT^\#] < \infty$. In this case we use Lemma 2.5 and obtain Λ and Γ such that (3.26) holds. Then (3.25) follows from (3.22).

(b) $\kappa^-[I - TT^\#] = \infty$. In this case we apply Lemma 2.5 to obtain Λ and Γ such that (3.25) holds. Then (3.26) is a consequence of (3.22).

Now, with Λ and Γ satisfying (3.25) and (3.26) define $T_r \in \mathscr{L}(\tilde{\mathscr{H}}_1, \mathscr{H}_2)$ an extension of T by the formula (3.4) with $\Delta = 0$.

Then remark that the factorizations (3.11) and (3.12) still hold true and these imply, via an argument of Pontryagin Lemma type, that (3.6) and (3.25) hold, hence T_r is a solution of Problem $(*)_r$.

The statement concerning the parametrization formula is a direct consequence of Lemma 3.1. ∎

There exists a problem which is similar to Problem $(*)_r$, concerning column extensions with prescribed negative signatures of defect. Let $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}'_2$ be Krein spaces, $\tilde{\mathscr{H}}_2 = \mathscr{H}_2[+]\mathscr{H}'_2$, $T \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ and $\kappa_1, \kappa_2 \in \tilde{\mathbb{N}}$ be given. The problem is

Determine all the operators $T_c \in \mathscr{L}(\mathscr{H}_1, \tilde{\mathscr{H}}_2)$ *with* $P^{\tilde{\mathscr{H}}_2}_{\mathscr{H}_2} T_c = T$,
$\kappa^-[I - T_c^\# T_c] = \kappa_1$, *and* $\kappa^-[I - T_c T_c^\#] = \kappa_2$. $\qquad (*)_c$

Clearly, Problem $(*)_c$ and Problem $(*)_r$ are equivalent by passing to the adjoint, thus the results which were obtained for Problem $(*)_r$ can be translated for Problem $(*)_c$. Since we will need precise formulation we state however the corresponding results.

From now on we assume $\mathscr{H}'_2$ is a Hilbert space. Fix J_1 and J_2 f.s.'s on $\mathscr{H}_1$ and $\mathscr{H}_2$. $\tilde{J}_2 = J_2 \oplus I$ is the fixed f.s. on $\tilde{\mathscr{H}}_2$. With respect to these will be considered defect operators, defect spaces, etc., as in Section 2.

3.5. Lemma. *Let* $T_c \in \mathscr{L}(\mathscr{H}_1, \tilde{\mathscr{H}}_2)$ *be such that* $P^{\tilde{\mathscr{H}}_2}_{\mathscr{H}_2} T_c = T$. *If we suppose*

$$P_{\overline{\mathscr{R}(D_T)}} \mathscr{R}(T_c) \subseteq \mathscr{R}(D_T) \tag{3.27}$$

then there exist uniquely determined $\Lambda \in \mathscr{L}(\ker D_T, \mathscr{H}'_2)$, $\Gamma \in \mathscr{L}(\mathscr{D}_T, \ker \Lambda^*)$, *and* $\Delta \in \mathscr{L}(\mathscr{D}_T, \mathscr{R}(\Lambda))$ *such that, with respect to the decompositions*

$$\mathscr{H}_1 = \mathscr{D}_T \oplus \ker D_T, \qquad \mathscr{H}'_2 = \ker \Lambda^* \oplus \mathscr{R}(\Lambda) \tag{3.28}$$

we have

$$P^{\tilde{\mathscr{H}}_2}_{\mathscr{H}'_2} T_c = \begin{bmatrix} \Gamma D_T & 0 \\ \Delta D_T & \Lambda \end{bmatrix}. \tag{3.29}$$

Moreover, if $\kappa^-[I - T_c^\# T_c] < \infty$ *then there exists a unique operator* $U(T_c)$ *such that*

$$\begin{gathered} U(T_c)\colon \mathscr{D}_{T_c} \to \mathscr{D}_\Gamma[+][-\mathscr{R}(\Lambda)] \\ U(T_c) D_{T_c} = \begin{bmatrix} D_\Gamma D_T & 0 \\ \Delta D_T & \Lambda \end{bmatrix} \end{gathered} \tag{3.30}$$

and the following identity holds

$$\kappa^-[I - T_c T_c^\#] = \kappa^-[I - TT^\#] + \kappa^-[I - \Gamma\Gamma^\#] + \operatorname{rank}(\Lambda). \tag{3.31}$$

In particular, $\kappa^-[I - T_c T_c^\#]$ *and* $\kappa^-[I - TT^\#]$ *are simultaneously finite and in this case there exists a unique unitary operator* $U_*(T_c)$ *such that*

$$U_*(T_c)\colon \mathscr{D}_{T_c^*} \to \mathscr{D}_{T^*}[+][\mathscr{R}(\Lambda^*) \oplus \mathscr{R}(\Lambda^*)][+]\mathscr{D}_{\Gamma^*}$$

$$U_*(T_c) D_{T_c^*} = \begin{bmatrix} D_{T^*} & 0 & -J_{T^*} L_T \Delta^* & -J_{T^*} L_T \Gamma^* \\ 0 & -P^{\mathscr{H}_1}_{\mathscr{R}(\Lambda^*)} J_1 T^* & \frac{1}{2}\Lambda^{-1} E_* & -\Lambda^{-1}\Delta J_T \Gamma^* \\ 0 & 0 & \Lambda^* & 0 \\ 0 & 0 & 0 & D_{\Gamma^*} \end{bmatrix}, \tag{3.32}$$

where

$$E_* = I - \Delta J_T \Delta^* - \Lambda P^{\mathscr{H}_1}_{\mathscr{R}(\Lambda^*)} J_1 \Lambda^*.$$

3.6. COROLLARY. *Let* $T_c \in \mathscr{L}(\mathscr{H}_1, \tilde{\mathscr{H}}_2)$ *be such that* $P^{\tilde{\mathscr{H}}_2}_{\mathscr{H}_2} T_c = T$ *and* (3.27) *holds. Then*

$$\kappa^-[I - T_c^\# T_c] + \kappa^-[I - TT^\#] = \kappa^-[I - T_c T_c^\#] + \kappa^-[I - T^\# T]. \tag{3.33}$$

3.7. THEOREM. *Assume* $\mathscr{H}'_2$ *is a Hilbert space. Then Problem* $(*)_c$ *has solutions if the conditions hold,*

$$\kappa_1+\kappa^-[I-TT^\#]=\kappa_2+\kappa^-[I-T^\# T], \tag{3.34}$$

$$\kappa^-[I-T^\# T]\leqslant\kappa_1\leqslant\kappa^-[I-T^\# T]+\kappa^{+0}, \tag{3.35}$$

$$\kappa^-[I-TT^\#]\leqslant\kappa_2\leqslant\kappa^-[I-TT^\#]+\kappa^{+0}, \tag{3.36}$$

where

$$\kappa^{+0}=\min\{\dim\mathscr{H}'_2, \kappa^+[I-T^\# T]+\kappa^0[I-T^\# T]\}.$$

Moreover, if the conditions (3.34)–(3.36) *hold, then the formula* (3.29) *establishes a bijective correspondence between a subset of solutions T_c of Problem $(*)_c$ and the set of triplets $\{\Lambda, \Gamma, \Delta\}$, where $\Lambda\in\mathscr{L}(\ker D_T, \mathscr{H}'_2)$ and $\Gamma\in\mathscr{L}(\mathscr{D}_T, \ker\Lambda^*)$ satisfy*

$$\kappa_1=\kappa^-[I-\Gamma\Gamma^\#]+\operatorname{rank}(\Lambda) \tag{3.37}$$

and $\Delta\in\mathscr{L}(\mathscr{D}_T, \mathscr{R}(\Lambda))$ is arbitrary.

3.8. *Remark.* The technical conditions (3.2) and (3.27) are fulfilled in many particular cases (see [11]). For instance, (3.2) holds if $\kappa^-[I-T_rT_r^\#]=0$ and similarly, (3.27) holds if $\kappa^-[I-T_c^\# T_c]=0$. They also hold if either $I-T_rT_r^\#$ or $I-T_c^\# T_c$ have closed range, in particular these are true in case the ambiental spaces have finite dimensions. However, even for $\kappa^-[I-T_rT_r^\#]$ finite, (3.2) is not true in general.

A simple example is the following: Let $\mathscr{H}$ be an infinite dimensional Hilbert space, $T\in\mathscr{L}(\mathscr{H})$ be a contraction such that $\mathscr{R}(I-T^*T)$ is dense but not closed. Consider $x\in\mathscr{H}-\mathscr{R}(D_{T^*})$ and take the row extension $T_r\in\mathscr{L}(\mathscr{H}\oplus\mathbb{C}, \mathscr{H})$

$$T_r=[T\quad x].$$

Then $\kappa^-(I-T_rT_r^*)=1$ but x cannot be factored by D_{T^*}.

4. Extensions with Data on a Degenerate Subspace

In view of our method of solving Problem $(*)$ (see Section 5) we need to solve a certain problem with data on a degenerate subspace of a Krein space. This is explained by the fact that, for instance, in the identification of the defect space $\mathscr{D}_{T_r}$ of a row extension T_r of a given operator T it appears the Krein space $[\mathscr{L}\oplus\mathscr{L}]$ (see the notation at the beginning of Section 3), where $\mathscr{L}$ is a certain Hilbert space. For this purpose we start with some results concerning the operator equation

$$B^*X+X^*B=C \tag{4.1}$$

considered in [17, 19].

Let $B \in \mathscr{L}(\mathscr{H}, \mathscr{G})$ and $C \in \mathscr{L}(\mathscr{H})$, $C = C^*$ be two given operators, where $\mathscr{H}$ and $\mathscr{G}$ are Hilbert spaces. With respect to the decompositions

$$\mathscr{H} = \overline{\mathscr{R}(B^*)} \oplus \ker B, \qquad \mathscr{G} = \overline{\mathscr{R}(B)} \oplus \ker B^* \tag{4.2}$$

we have the representations

$$B = \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{22} \end{bmatrix} \tag{4.3}$$

with $C_{11} = C_{11}^*$ and $C_{22} = C_{22}^*$. The following lemma is a reformulation of results from [17, 19], so we omit the proof.

4.1. Lemma. *If B has closed range then the operator equation* (4.1) *has solutions if and only if*

$$C \ker B \subseteq \mathscr{R}(B^*). \tag{4.4}$$

Moreover, if (4.4) *holds, then the formula*

$$X = \begin{bmatrix} B_{11}^{*\,-1}(\frac{1}{2}C_{11} + iS) & B_{11}^{*\,-1} C_{12} \\ Y & Z \end{bmatrix} \tag{4.5}$$

with respect to the decompositions (4.2), *establishes a bijective correspondence between the set of solutions of Eq.* (4.1) *and the set of triplets* $\{S, Y, Z\}$ *with* $S \in \mathscr{L}(\mathscr{R}(B^*))$, $S = S^*$ *and* $Y \in \mathscr{L}(\mathscr{R}(B^*), \ker B^*)$, $Z \in \mathscr{L}(\ker B, \ker B^*)$ *are arbitrary.*

As a first consequence we state the following result.

4.2. Lemma. *With previous notation and the assumption that B has closed range we have*

$$\begin{aligned}&\{\kappa^-(C - B^*X - X^*B) \mid X \in \mathscr{L}(\mathscr{H}, \mathscr{G})\}\\ &\quad = \{\kappa \in \tilde{\mathbb{N}} \mid \kappa^-(P^{\mathscr{H}}_{\ker B} C \mid \ker B) \leqslant \kappa \leqslant \kappa^-(P^{\mathscr{H}}_{\ker B} C \mid \ker B) + \operatorname{rank} B\}.\end{aligned}$$

Proof. By Lemma 4.1 we obtain

$$\begin{aligned}&\{C - B^*X - X^*B \mid X \in \mathscr{L}(\mathscr{H}, \mathscr{G})\}\\ &\quad = \{D \in \mathscr{L}(\mathscr{H}) \mid D = D^*, P^{\mathscr{H}}_{\ker B} D \mid \ker B = P^{\mathscr{H}}_{\ker B} C \mid \ker B\},\end{aligned}$$

and then use Lemma 2.4. ∎

Now we can return to the main concern of this section. Let us consider the Krein spaces $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}'_2$, and the operators $A \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$, $B \in \mathscr{L}(\mathscr{H}_1, \mathscr{L})$, where $\mathscr{L}$ is a Hilbert space.

Define the Krein space

$$\tilde{\mathscr{K}} = \mathscr{H}_2[+][\mathscr{L} \oplus \mathscr{L}][+]\mathscr{H}'_2 \tag{4.6}$$

and consider J_1, J_2, J'_2, and J_0 (defined at (3.1)) f.s.'s on $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{H}'_2$, and respectively $[\mathscr{L} \oplus \mathscr{L}]$. Then

$$\tilde{J} = J_2 \oplus J_0 \oplus J'_2 \tag{4.7}$$

is f.s. on $\tilde{\mathscr{K}}$. For arbitrary $U \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}'_2)$ and $V \in \mathscr{L}(\mathscr{H}, \mathscr{L})$ define an operator $T_{U,V} \in \mathscr{L}(\mathscr{H}_1, \tilde{\mathscr{K}})$ by

$$T_{U,V} = [A \quad B \quad V \quad U]^t, \tag{4.8}$$

where "t" denotes the matrix transpose. The main result of this section is the following.

4.3. PROPOSITION. *With previous notation and assuming that B has closed range we have*

$$\begin{aligned}\{\kappa^-[I - T^{\#}_{U,V}T_{U,V}] \mid U \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}'_2), V \in \mathscr{L}(\mathscr{H}_1, \mathscr{L})\}\\ = \{\kappa \in \bar{\mathbb{N}} \mid \max\{0, \kappa^-(S) - \kappa^-[\mathscr{H}'_2]\\ \leqslant \kappa \leqslant \kappa^-(S) + \min\{\kappa^+[\mathscr{H}'_2], \kappa^+(S) + \kappa^0(S) + \operatorname{rank} B\}\},\end{aligned}$$

where $S = P^{\mathscr{H}_1}_{\ker B}(J_1 - A^*J_2A) \mid \ker B$ *and we used the convention that if both of* $\kappa^-(S)$ *and* $\kappa^-[\mathscr{H}'_2]$ *are infinite, then* $\kappa^-(S) - \kappa^-[\mathscr{H}'_2] = 0$.

Proof. With $\tilde{J}$ defined by (4.7) we have

$$J_1 - T^*_{U,V}\tilde{J}T_{U,V} = J_1 - A^*J_2A - U^*J'_2U - B^*V - V^*B$$

and now the result follows from Lemmas 4.2 and 2.5. ∎

In a certain special case we will need the description of all extensions $T_{U,V}$. For this purpose it is convenient to represent $T_{U,V}$ by

$$T_{U,V} = \begin{bmatrix} A_1 & 0 & V_1 & U_1 \\ A_2 & B & V_2 & U_2 \end{bmatrix}^t \tag{4.9}$$

with respect to the decompositions $\mathscr{H}_1 = \ker B \oplus \overline{\mathscr{R}(B^*)}$ and (4.6).

4.4. LEMMA. *Suppose* $\mathscr{H}_1$ *is a Hilbert space, B is surjective and moreover,*

$$\kappa^-[I - A_1^{\#}A_1] = \kappa^-[\mathscr{H}'_2] < \infty. \tag{4.10}$$

Then the formulae

$$\left.\begin{aligned} V_1 &= -B^{*-1}(A_2^* J_2 A_1 + R^* J_2' G D_{A_1} + P^{1/2} C^* D_G D_{A_1}) \\ V_2 &= B^{*-1}(\tfrac{1}{2}(I - A_2^* J_2 A_2 - R^* J_2 R - P) + iS) \\ U_1 &= G D_{A_1} \\ U_2 &= R \end{aligned}\right\} \tag{4.11}$$

establish a bijective correspondence between the set of all contractive extensions $T_{U,V}$ *of the form* (4.9) *and the set of* 5-*tuples* $\{G, P, S, C, R\}$ *such that*

$$\left.\begin{array}{lll} G: & \mathscr{D}_{A_1} \to \mathscr{H}'_2, & \textit{Pontryagin space contraction} \\ P: & \mathscr{R}(B^*) \to \mathscr{R}(B^*), & P \geqslant 0 \\ S: & \mathscr{R}(B^*) \to \mathscr{R}(B^*), & S = S^* \\ C: & \overline{\mathscr{R}(P)} \to \mathscr{D}_G, & \textit{a Hilbert space contraction} \\ R: & \mathscr{R}(B^*) \to \mathscr{H}'_2 & \textit{arbitrary.} \end{array}\right\} \tag{4.12}$$

Proof. Assume that $T_{U,V}$ given by (4.9) is contractive. Then $[A_1 \quad U_1]'$: $\ker B \to \mathscr{H}_2[+]\mathscr{H}'_2$ is contractive and from Lemma 3.1 we deduce $U_1 = GD_{A_1}$ with $G: \mathscr{D}_{A_1} \to \mathscr{H}'_2$ a Pontryagin space contraction. Further on, we have

$$I - T_{U,V}^* \tilde{J} T_{U,V} = \begin{bmatrix} D_{A_1} D_G^2 D_{A_1} & X \\ X^* & P \end{bmatrix} \geqslant 0,$$

where

$$P = I - A_2^* J_2 A_2 - U_2^* J_2' U_2 - B^* V_2 - V_2^* B$$

and

$$X = -A_1^* J_2 A_2 - D_{A_1} G^* J_2 U_2 - V_1^* B.$$

It follows $P \geqslant 0$ and $X = D_{A_1} D_G C P^{1/2}$ for a contraction $C \in \mathscr{L}(\mathscr{R}(P), \mathscr{D}_G)$. Take $U_2 = R \in \mathscr{L}(\mathscr{R}(B^*), \mathscr{H}'_2)$ arbitrary.

The form of V_2 is now clear and for V_1 we use Lemma 4.1. ▮

5. A Lifting Theorem

Let $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{H}'_1$, and $\mathscr{H}'_2$ be Krein spaces and denote $\tilde{\mathscr{H}}_1 = \mathscr{H}_1 [+] \mathscr{H}'_1$, $\tilde{\mathscr{H}}_2 = \mathscr{H}_2 [+] \mathscr{H}'_2$. Consider two operators $T_r \in \mathscr{L}(\tilde{\mathscr{H}}_1, \mathscr{H}_2)$ and $T_c \in \mathscr{L}(\mathscr{H}_1, \tilde{\mathscr{H}}_2)$ such that

$$T_r \mid \mathscr{H}_1 = P_{\mathscr{H}_2}^{\tilde{\mathscr{H}}_2} T_c = T \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2) \tag{5.1}$$

and, for given κ_1, $\kappa_2 \in \tilde{\mathbb{N}}$, the identities

$$\kappa^-[I - T_c^\# T_c] = \kappa_1 \tag{5.2}$$

$$\kappa^-[I - T_r T_r^\#] = \kappa_2 \tag{5.3}$$

hold.

We consider the following problem of lifting of operators with prescribed negative signature of defects:

Determine all operators $\tilde{T} \in \mathscr{L}(\tilde{\mathscr{H}}_1, \tilde{\mathscr{H}}_2)$ *such that* $P_{\mathscr{H}_2}^{\tilde{\mathscr{H}}_2} \tilde{T} = T_r$, $\tilde{T} \mid \mathscr{H}_1 = T_c$, $\kappa^-[I - \tilde{T}^\# \tilde{T}] = \kappa_1$ *and* $\kappa^-[I - \tilde{T}\tilde{T}^\#] = \kappa_2$. $\quad (*)$

In this section we solve this problem in the case when $\mathscr{H}'_1$ and $\mathscr{H}'_2$ are Hilbert spaces, (3.2) and (3.27) hold, and κ_1, κ_2 are finite. These hypotheses will be assumed throughout this section.

Let us fix J_1 and J_2 two f.s.'s on $\mathscr{H}_1$ and respectively $\mathscr{H}_2$, $\tilde{J}_1 = J_1 \oplus I$ and $\tilde{J}_2 = J_2 \oplus I$ will be the fixed f.s.'s on $\tilde{\mathscr{H}}_1$ and $\tilde{\mathscr{H}}_2$.

With respect to these f.s.'s will be considered defect operators, defect spaces, etc. (see Section 2).

By (5.1) T_r is a row extension of T and since κ_2 is finite and (5.3) holds, application of Lemma 3.1 yields uniquely determined operators $\Lambda_1 \in \mathscr{L}(\mathscr{H}'_1, \ker D_{T^*})$ of finite rank, $\Gamma_1 \in \mathscr{L}(\ker \Lambda_1, \mathscr{D}_{T^*})$ and $\Delta_1 \in \mathscr{L}(\mathscr{R}(\Lambda_1^*), \mathscr{D}_{T^*})$ such that

$$T_r \mid \mathscr{H}'_1 = \begin{bmatrix} D_{T^*}\Gamma_1 & D_{T^*}\Delta_1 \\ 0 & \Lambda_1 \end{bmatrix}. \tag{5.4}$$

Similarly, T_c is a column extension of T and from (5.2) and the finiteness of κ_1, application of Lemma 3.5 yields uniquely determined operators $\Lambda_2 \in \mathscr{L}(\ker D_T, \mathscr{H}'_2)$ of finite rank, $\Gamma_2 \in \mathscr{L}(\mathscr{D}_T, \ker \Lambda_2^*)$ and $\Delta_2 \in \mathscr{L}(\mathscr{D}_T, \mathscr{R}(\Lambda_2))$ such that

$$P_{\mathscr{H}'_2}^{\tilde{\mathscr{H}}_2} T_c = \begin{bmatrix} \Gamma_2 D_T & 0 \\ \Delta_2 D_T & \Lambda_2 \end{bmatrix}. \tag{5.5}$$

First we find necessary and sufficient conditions for the solvability of Problem $(*)$.

5.1. THEOREM. *With the notation and assumptions stated above, Problem* $(*)$ *has solutions if and only if the following conditions hold*

(i) $P_{\ker D_{T^*}}^{\mathscr{H}_2} T J_1 \mathscr{R}(\Lambda_2^*) = \mathscr{R}(\Lambda_1)$,

(*or, equivalently*, $P_{\ker D_T}^{\mathscr{H}_1} T^* J_2 \mathscr{R}(\Lambda_1) = \mathscr{R}(\Lambda_2^*)$).

(ii) $\kappa^-[I - T_r^\# T_r] = \kappa_1$.

(iii) $\kappa^-[I - T_c T_c^\#] = \kappa_2$.

Proof. Let $\tilde{T}$ be a solution of Problem (*). Regarding $\tilde{T}$ as a row extension of T_c, it follows from [11, Lemma 2.1] that there exists a uniquely determined contraction $\Theta' \in \mathscr{L}(\mathscr{H}_1, \mathscr{D}_{T_c^*})$ such that

$$\tilde{T} = [T_c \quad D_{T_c^*}\Theta'] \tag{5.6}$$

and

$$\kappa_2 = \kappa^-[I - \tilde{T}\tilde{T}^\#] = \kappa^-[I - T_c T_c^\#]$$

hence the condition (iii) is fulfilled.

Further on, define an operator $\Theta \in \mathscr{L}(\mathscr{H}'_1, \mathscr{D}_{T^*}[+][\mathscr{R}(A_2^*) \oplus \mathscr{R}(A_2^*)][+]\mathscr{D}_{\Gamma_2^*})$ by means of the identity

$$\Theta' = U_*(T_c)^*\Theta, \tag{5.7}$$

where $U_*(T_c)$ is the unitary operator defined by (3.32). Θ is also a contraction. Let $\Theta = (\Theta_{ij} \mid 1 \leqslant i \leqslant 4,\ 1 \leqslant j \leqslant 2)$ be the matrix representation of Θ with respect to the decomposition $\ker A_1 \oplus \mathscr{R}(A_1^*)$ of $\mathscr{H}'_1$.

Then

$$\tilde{T} \mid \mathscr{H}'_1 = D_{T_c^*}\Theta' = D_{T_c^*}U_*(T_c)^*\Theta$$

$$= \begin{bmatrix} D_{T^*}\Theta_{11} & D_{T^*}\Theta_{12} \\ -P^{\mathscr{H}_2}_{\ker D_{T^*}} TJ_1\Theta_{21} & -P^{\mathscr{H}_2}_{\ker D_{T^*}} TJ_1\Theta_{22} \\ * & * \\ * & * \end{bmatrix}, \tag{5.8}$$

where the entries marked by "*" play no role for the moment. From the uniqueness of the representation (5.4) and (5.8) we obtain $\Theta_{11} = \Gamma_1$, $\Theta_{12} = A_1$, $\Theta_{21} = 0$, and

$$-P^{\mathscr{H}_2}_{\ker D_{T^*}} TJ_1\Theta_{22} = A_1 \tag{5.9}$$

which implies

$$P^{\mathscr{H}_2}_{\ker D_{T^*}} TJ_1\mathscr{R}(A_2^*) \supseteq \mathscr{R}(A_1). \tag{5.10}$$

On the other hand, regarding $\tilde{T}$ this time as a column extension of T_r, by [11, Lemma 2.2] it follows that there exists a uniquely determined operator $\Psi' \in \mathscr{L}(\mathscr{D}_{T_r}, \mathscr{H}'_2)$, $\Psi'^\#$ being a contraction, such that

$$T = [T_r \quad \Psi' D_{T_r}]' \tag{5.11}$$

and

$$\kappa_1 = \kappa^-[I - \tilde{T}^\# \tilde{T}] = \kappa^-[I - T_r^\# T_r],$$

hence the condition (ii) is also fulfilled.

Further on, define an operator $\Psi \in \mathscr{L}(\mathscr{D}_T[+][\mathscr{R}(\Lambda_1) \oplus \mathscr{R}(\Lambda_1)][+]\mathscr{D}_{\Gamma_1}, \mathscr{H}'_2)$ by means of the equality

$$\Psi U(T_r) = \Psi', \tag{5.12}$$

where $U(T_r)$ is the unitary operator defined by (3.7). $\Psi^{\#}$ is also a contraction. If $(\Psi_{ij}/1 \leqslant i \leqslant 2,\ 1 \leqslant j \leqslant 4)$ is the matrix representation of Ψ with respect to the decomposition $\mathscr{H}'_2 = \ker \Lambda_2^* \oplus \mathscr{R}(\Lambda_2)$, then

$$\begin{aligned} P^{\mathscr{H}_2}_{\mathscr{H}'_2} \tilde{T} &= \Psi' D_{T_r} = \Psi U(T_r) D_{T_r} \\ &= \begin{bmatrix} \Psi_{11} D_T & -\Psi_{12} P^{\mathscr{H}_2}_{\mathscr{R}(\Lambda_1)} J_2 T | \ker D_T & * & * \\ \Psi_{21} D_T & -\Psi_{22} P^{\mathscr{H}_2}_{\mathscr{R}(\Lambda_1)} J_2 T | \ker D_T & * & * \end{bmatrix}. \end{aligned} \tag{5.13}$$

From (5.13) and the uniqueness of the representation (5.5) we obtain $\Psi_{11} = \Gamma_2$, $\Psi_{21} = \Delta_2$, $\Psi_{12} = 0$, and

$$-\Psi_{22} P^{\mathscr{H}_2}_{\mathscr{R}(\Lambda_1)} J_2 T | \ker D_T = \Lambda_2 \tag{5.14}$$

and this implies

$$P^{\mathscr{H}_1}_{\ker D_T} T^* J_2 \mathscr{R}(\Lambda_1) \supseteq \mathscr{R}(\Lambda_2^*). \tag{5.15}$$

Taking into account Remark 2.6, from (5.10) and (5.15) we obtain the condition (i).

Conversely, assuming that the conditions (i)–(iii) are fulfilled, let us take $\Theta_{11} = \Gamma_1$, $\Theta_{12} = \Delta_1$, $\Theta_{21} = 0$, and

$$\Theta_{22} = -P^{\mathscr{H}_1}_{\ker D_T} T^* J_2 \Lambda_1 \in \mathscr{L}(\mathscr{R}(\Lambda_1^*), \mathscr{R}(\Lambda_2^*))$$

(the fact that the definition of Θ_{22} is correct follows from condition (i)). Moreover, condition (i) implies, via Remark 2.6,

$$\operatorname{rank}(\Lambda_1) = \operatorname{rank}(\Lambda_2). \tag{5.16}$$

Now from (3.6), (3.31), (5.16), and condition (ii) we obtain

$$\kappa^-[I - \Gamma_1^{\#} \Gamma_1] = \kappa^-[I - \Gamma_2 \Gamma_2^{\#}] < \infty$$

and then remark that we can apply Proposition 4.3 in order to obtain a contraction

$$\Theta : \mathscr{H}'_1 \to \mathscr{D}_{T^*}[+][\mathscr{R}(\Lambda_2^*) \oplus \mathscr{R}(\Lambda_2^*)][+]\mathscr{D}_{\Gamma_2^*}$$

$$\Theta = \begin{bmatrix} \Gamma_1 & \Delta_1 \\ 0 & -P^{\mathscr{H}_1}_{\ker D_T} T^* J_2 \Lambda_1 \\ \Theta_{31} & \Theta_{32} \\ \Theta_{41} & \Theta_{42} \end{bmatrix}, \tag{5.17}$$

where Θ is represented with respect to the decomposition $\mathscr{H}'_1 = \ker \Lambda_1 \oplus \mathscr{R}(\Lambda_1^*)$. With this Θ define Θ' by means of (5.7) and $\tilde{T}$ by (5.6). We have $\tilde{T} \mid \mathscr{H}_1 = T_c$ and $P^{\tilde{\mathscr{H}}_2}_{\mathscr{H}} \tilde{T} = T_r$ and since Θ' is also a contraction, we use [11, Lemma 2.1] and obtain $\kappa^-[I - \tilde{T}^\# \tilde{T}] = \kappa_1$, $\kappa^-[I - \tilde{T}\tilde{T}^\#] = \kappa_2$, hence $\tilde{T}$ is a solution of Problem $(*)$. ∎

5.2. *Remark.* During the proof of the converse implication in Theorem 5.2 we did not use at all the condition (iii). As a matter of fact, the conditions (i), (ii), and (iii) are not independent, more precisely, it is easy to see that (i) and (ii) imply (iii), and (i) and (iii) imply (ii). On the other hand, if $\mathscr{H}_1$ and $\mathscr{H}_2$ are Pontryagin spaces, then (ii) is equivalent with (iii) and they read simply

$$\kappa_1 + \kappa^-[\mathscr{H}_2] = \kappa_2 + \kappa^-[\mathscr{H}_1] \tag{5.18}$$

which, taking account of (2.13), is obviously a necessary condition for the existence of solutions of Problem $(*)$.

5.3. *Remark.* Assuming $\mathscr{H}_1$ and $\mathscr{H}_2$ Hilbert spaces, i.e., J_1 and J_2 are the identity operators, one notes that $T \mid \ker D_T \in \mathscr{L}(\ker D_T, \ker D_{T^*})$ is unitary. In this case the condition (i) reads $T\mathscr{R}(\Lambda_2^*) = \mathscr{R}(\Lambda_1)$ and by (5.18), the condition (ii) (or, equivalently (iii)) means $\kappa_1 = \kappa_2$. It follows that Theorem 1.1 can be deduced from Theorem 5.1.

The proof of Theorem 5.1 enables us to solve completely Problem $(*)$ (in the case $\mathscr{H}'_1$ and $\mathscr{H}'_2$ are Hilbert spaces and κ_1, κ_2 are finite). In the following, for a solution $\tilde{T}$ of Problem $(*)$ we denote $Z = P^{\tilde{\mathscr{H}}_2}_{\mathscr{H}'_2} \tilde{T} \mid \mathscr{H}'_1$ and consider the matrix representation $Z = (Z_{ij})_{1 \leqslant i, j \leqslant 2}$ with respect to the decompositions $\mathscr{H}'_1 = \ker \Lambda_1 \oplus \mathscr{R}(\Lambda_1^*)$ and $\mathscr{H}'_2 = \ker \Lambda_2 \oplus \mathscr{R}(\Lambda_2^*)$.

5.4. THEOREM. *Assume that the conditions* (i)–(iii) *from Theorem* 5.1 *are fulfilled. Then, with the notation stated above, the formulae*

$$\left.\begin{aligned}
Z_{11} &= -\Gamma_2 L_T^* J_{T^*} \Gamma_1 + D_{\Gamma_2^*} G D_{\Gamma_1},\\
Z_{12} &= -\Gamma_2 L_T^* J_{T^*} \Delta_1 + \Gamma_2 J_T \Delta_2^* \Lambda_2^{*-1} T^* J_2 \Lambda_1 + D_{\Gamma_2^*} R,\\
Z_{21} &= -\Delta_2 L_T^* J_{T^*} \Gamma_1 + \Lambda_2 J_1 T^* \Lambda_1^{*-1}\\
&\quad \times (\Delta_1^* J_{T^*} \Gamma_1 + R^* J_{\Gamma_2^*} G D_{\Gamma_1} + P^{1/2} C^* D_G D_{\Gamma_1}),\\
Z_{22} &= -\Delta_2 L_T^* J_{T^*} \Delta_1 + \tfrac{1}{2}(\Lambda_2 J_1 \mid \mathscr{R}(\Lambda_2^*) + \Delta_2 J_T \Delta_2^* \Lambda_2^{*-1} - \Lambda_2^{*-1})\\
&\quad \times P^{\mathscr{H}_1}_{\mathscr{R}(\Lambda_2^*)} T^* J_2 \Lambda_1 - \Lambda_2 J_1 T^* \Lambda_1^{*-1}\\
&\quad \times (\tfrac{1}{2}(I - \Delta_1^* J_{T^*} \Delta_1 - R^* J_{\Gamma_2^*} R - P) + iS)
\end{aligned}\right\} \tag{5.19}$$

establish a bijective correspondence between the set of solutions $\tilde{T}$ of Problem (∗) *and the set of* 5-*tuples* $\{G, P, S, C, R\}$ *such that*

$$\left.\begin{array}{lll} G\colon & \mathscr{D}_{\Gamma_1} \to \mathscr{D}_{\Gamma_2^*}, & \text{a Pontryagin space contraction,} \\ P\colon & \mathscr{R}(\Lambda_1^*) \to \mathscr{R}(\Lambda_1^*), & P \geqslant 0, \\ S\colon & \mathscr{R}(\Lambda_1^*) \to \mathscr{R}(\Lambda_1^*), & S = S^*, \\ C\colon & \mathscr{R}(P) \to \mathscr{D}_G, & \text{a Hilbert space contraction,} \\ R\colon & \mathscr{R}(\Lambda_1^*) \to \mathscr{D}_{\Gamma_2^*}, & \text{arbitrary.} \end{array}\right\} \tag{5.20}$$

Proof. From the proof of Theorem 5.1 it follows that we have to determine the set of contractions Θ as in (5.17), then remark that we can use Lemma 4.4 and obtain

$$\left.\begin{array}{l} \Theta_{31} = -J_1 T^* \Lambda_1^{*\,-1}(\Delta_1^* J_{T^*} \Gamma_1 + R^* J_{T^*} G D_{\Gamma_1} + P^{1/2} C^* D_G D_{\Gamma_1}), \\ \Theta_{32} = -J_1 T^* \Lambda_1^{*\,-1}(\tfrac{1}{2}(I - \Delta_1^* J_{T^*} \Delta_1 - R^* J_{\Gamma_2^*} R - P) + iS), \\ \Theta_{41} = G D_{\Gamma_1}, \qquad \Theta_{42} = R, \end{array}\right\} \tag{5.21}$$

where the parameters are described by (5.20). Now, the formulae (5.19) can be easily obtained from (5.21), (5.17), and (5.8). ∎

5.5. *Remark.* From Theorem 5.4 we easily deduce a criterion when Problem (∗) has a unique solution. Of course, we start with nontrivial spaces $\mathscr{H}_1'$ and $\mathscr{H}_2'$ and we obtain:

Problem (∗) has a unique solution if and only if Λ_1 and Λ_2 are null operators and, either $\Gamma_2^{\#}$ is an isometry and $\kappa^-[I - TT^{\#}] = \kappa_2$, or Γ_1 is an isometry and $\kappa^-[I - T^{\#}T] = \kappa_1$.

5.6. *Remark.* The analysis of Problem (∗) that we performed for the case of Hilbert spaces $\mathscr{H}_1'$ and $\mathscr{H}_2'$ can be adapted also when these spaces are of Pontryagin type. The corresponding results are slightly more complicated and we will make them explicit only when appropriate applications will be considered.

6. Applications

In this section we consider two applications of the results in Section 5. Other applications will be presented elsewhere.

A. *Selfadjoint Extensions of Symmetric Operators*

Let $\mathscr{H}$ be a Hilbert space and $\mathscr{G}$ a subspace of $\mathscr{H}$. Let also $T_c \in \mathscr{L}(\mathscr{G}, \mathscr{H})$ be a symmetric operator, i.e.,

$$(T_c x, y) = (x, T_c y), \qquad x, y \in \mathscr{G}, \tag{6.1}$$

such that $\kappa^-(I - T_c^* T_c) = \kappa < \infty$ (equivalently, $\kappa^-(I - T_c T_c^*) = \kappa < \infty$).

The problem is

Determine all selfadjoint operators $\tilde{T} \in \mathscr{L}(\mathscr{H})$ such that $\tilde{T} | \mathscr{G} = T_c$ and $\kappa^-(I - \tilde{T}^2) = \kappa$. $(*)_s$

This problem was considered in [18] as a generalization of a problem of M. G. Krein [15].

Let us represent T_c with respect to the decomposition $\mathscr{H} = \mathscr{G} \oplus \mathscr{G}^\perp$

$$T_c = [T \quad Y]'. \tag{6.2}$$

From (6.1) it follows that $T \in \mathscr{L}(\mathscr{G})$ is selfadjoint, $T = T^*$, and assume that $P_{\overline{\mathscr{R}(D_T)}} \mathscr{R}(Y^*) \subseteq \mathscr{R}(D_T)$. Then application of Lemma 3.5 yields uniquely determined operators $\Lambda \in \mathscr{L}(\ker D_T, \mathscr{G}^\perp)$ of finite rank, $\Gamma \in \mathscr{L}(\mathscr{D}_T, \ker \Lambda^*)$ and $\Delta \in \mathscr{L}(\mathscr{D}_T, \mathscr{R}(\Lambda))$, such that the operator $Y \in \mathscr{L}(\mathscr{G}, \mathscr{G}^\perp)$ is represented by

$$Y = \begin{bmatrix} \Gamma D_T & 0 \\ \Delta D_T & \Lambda \end{bmatrix} \tag{6.3}$$

with respect to the decompositions $\mathscr{G} = \mathscr{D}_T \oplus \ker D_T$ and $\mathscr{G}^\perp = \ker \Lambda^* \oplus \mathscr{R}(\Lambda)$. First we settle the solvability of Problem $(*)_s$.

6.1. PROPOSITION. *With the notation stated above, the following assertions are equivalent:*

(α) *Problem $(*)_s$ has at least one solution.*

(β) $T\mathscr{R}(\Lambda^*) = \mathscr{R}(\Lambda^*)$.

(γ) $\mathscr{R}(\Lambda^*) = \mathscr{L}_+ \oplus \mathscr{L}_-$, *where* $\mathscr{L}_\pm \subseteq \ker(I \mp T)$.

Proof. (α) ⇒ (β). If Problem $(*)_s$ has a solution, say $\tilde{T}$, then denoting $T_r = T_c^* \in \mathscr{L}(\mathscr{H}, \mathscr{G})$ it follows that $\tilde{T}$ is a solution of Problem $(*)$ corresponding to these T_c and T_r, hence (β) follows from Theorem 5.1.

(β) ⇒ (α). If (β) holds, then we define $\tilde{T}^{(0)} \in \mathscr{L}(\mathscr{H})$ by

$$\tilde{T}^{(0)} = \begin{bmatrix} T & Y^* \\ Y & Z^{(0)} \end{bmatrix}, \tag{6.4}$$

where $Z^{(0)} \in \mathscr{L}(\mathscr{G}^{\perp})$ has the matrix representation $(Z_{ij}^{(0)})_{1 \leqslant i,j \leqslant 2}$ with respect to the decomposition $\mathscr{G}^{\perp} = \ker(\Lambda^*) \oplus \mathscr{R}(\Lambda)$, given by

$$\left.\begin{aligned} Z_{11}^{(0)} &= -\Gamma T J_T \Gamma^* + D_{\Gamma^*}^2, \\ Z_{12}^{(0)} &= -\Gamma T J_T \Delta^* + \Gamma J_T \Delta^* \Lambda^{*-1} T \Lambda^*, \\ Z_{21}^{(0)} &= -\Delta T J_T \Gamma^* + \Lambda T \Lambda^{-1} \Delta J_T \Gamma^*, \\ Z_{22}^{(0)} &= -\Delta T J_T \Delta^* + \tfrac{1}{2}(\Lambda T \Lambda^* + \Lambda T \Lambda^{-1}(\Delta J_T \Delta^* - I) \\ &\quad + (\Delta J_T \Delta^* - I)\Lambda^{*-1} T \Lambda^*). \end{aligned}\right\} \tag{6.5}$$

From Theorem 5.4 it follows that $\tilde{T}^{(0)}$ is a solution for Problem $(*)$ corresponding to T_c and $T_r = T_c^*$ (indeed, (6.5) corresponds to the formulae in the statement of Theorem 5.4 with the choice $G = I$ and null P, R, S, and C) and it is readily verified that $\tilde{T}^{(0)}$ is selfadjoint, hence it is a solution for Problem $(*)_s$.

$(\beta) \Rightarrow (\gamma)$. This equivalence comes from the remark that $T \mid \ker D_T \in \mathscr{L}(\ker D_T)$ is a symmetry. ∎

6.2. *Remark.* Assuming that the condition (α) from Proposition 6.1 holds, by means of Theorem 5.4 one can obtain a "parametrization" of the set of solutions of Problem $(*)_s$. More precisely, denoting

$$\tilde{T} = \begin{bmatrix} T & Y^* \\ Y & Z \end{bmatrix}, \tag{6.6}$$

$Z = (Z_{ij})_{1 \leqslant i,j \leqslant 2}$ with respect to $\mathscr{G}^{\perp} = \ker(\Lambda^*) \oplus \mathscr{R}(\Lambda)$ and considering $Z^{(0)}$ defined by (6.5), the formulae

$$\left.\begin{aligned} Z_{11} &= Z_{11}^{(0)} + D_{\Gamma^*}(G - I) D_{\Gamma^*}, \\ Z_{12} &= Z_{12}^{(0)} + D_{\Gamma^*} R \\ Z_{21} &= Z_{12}^* \\ Z_{22} &= Z_{22}^{(0)} + \tfrac{1}{2} \Lambda T \Lambda^{-1}(R^* J_{\Gamma^*} R + P) - i \Lambda T \Lambda^{-1} S \end{aligned}\right\} \tag{6.7}$$

give the desired parametrization, where the parameters

$$\begin{aligned} &G\colon \mathscr{D}_{\Gamma^*} \to \mathscr{D}_{\Gamma^*}, && G = G^*,\ G J_{\Gamma^*} G \leqslant J_{\Gamma^*} \\ &P\colon \mathscr{R}(\Lambda) \to \mathscr{R}(\Lambda), && P \geqslant 0, \\ &S\colon \mathscr{R}(\Lambda) \to \mathscr{R}(\Lambda), && S = S^*, \\ &C\colon \mathscr{R}(P) \to \mathscr{D}_G, && C^* C \leqslant I, \\ &R\colon \mathscr{R}(\Lambda) \to \mathscr{D}_{\Gamma^*}, && \end{aligned}$$

satisfy, in addition, the operatorial equations

$$RA^{*-1}TA^* - GJ_{\Gamma^*}R = D_G CP^{1/2} \tag{6.9}$$

$$ATA^{-1}S + SA^{*-1}TA^* = \frac{1}{2i}(ATA^{-1}(R^*J_{\Gamma^*}R + P) - (R^*J_{\Gamma^*}R + P)A^{*-1}TA^*). \tag{6.10}$$

6.3. *Remark.* Let $dE(\lambda)$ denote the spectral measure of the selfadjoint operator $T \in \mathscr{L}(\mathscr{G})$. Using the representation (6.3) it is easy to prove the following assertions:

(i) The improper integral

$$\int_0^1 (1-\lambda)^{-1} Y\, dE(\lambda)\, Y^* \tag{6.11}$$

converges strongly (equivalently, weakly) if and only if $\mathscr{R}(A^*) \subseteq \ker(I + T)$.

(ii) The improper integral

$$\int_{-1}^0 (1+\lambda)^{-1} Y\, dE(\lambda)\, Y^* \tag{6.12}$$

converges strongly (equivalently, weakly) if and only if $\mathscr{R}(A^*) \subseteq \ker(I - T)$.

In particular, the criterion found in [18] (that is, if at least one of the integrals (6.11) or (6.12) converges, then Problem $(*)_s$ has solutions) can be obtained as a consequence of Proposition 6.1. Moreover, the same Proposition 6.1 shows that, in general, it is possible that both of the integrals (6.11) and (6.12) diverge and Problem $(*)_s$ still be solvable.

Finally, if, let us say, the integral (6.12) is convergent then Eq. (6.10) is equivalent with $S = 0$ and the equation (6.9) is easy to solve, hence the parametrization described in Remark 6.2 is explicit.

B. *Strictly Intertwining Dilatations*

Let us briefly recall the setting established in [11].

Consider $T_i \in \mathscr{L}(\mathscr{H}_i)$ contraction, where $\mathscr{H}_i$ is a Krein space and fix a f.s. J_i on $\mathscr{H}_i$, $i = 1, 2$. Denote by $(V_i, \mathscr{K}_i)$ the minimal isometric dilation of T_i (see [5, 10]) and consider the exhausting chain of subspaces $\{\mathscr{K}_i^{(n)}\}_{n \geqslant 0}$ of $\mathscr{K}_i$ defined by

$$\mathscr{K}_i^{(n)} = \begin{cases} \mathscr{H}_i, & n = 0, \\ \mathscr{H}_i [+] \underbrace{\mathscr{D}_{T_i} [+] \cdots [+] \mathscr{D}_{T_i}}_{n \text{ terms}}, & n \geqslant 1, \end{cases} \tag{6.13}$$

and the partial isometries

$$T_i^{(n)} = P^{\mathcal{K}_i}_{\mathcal{K}_i^{(n)}} V_i \,|\, \mathcal{K}_i^{(n)}. \tag{6.14}$$

Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be an operator such that

$$AT_1 = T_2 A \tag{6.15}$$

and for each integer $n \geqslant 0$ define the set of strictly intertwining dilations of order n by

$$\begin{aligned}\mathcal{S}^{(n)}(A; T_1, T_2) = \{A_n \in \mathcal{L}(\mathcal{K}_1^{(n)}, \mathcal{K}_2^{(n)}) \,|\, P^{\mathcal{K}_2^{(n)}}_{\mathcal{H}_2} A_n = A P^{\mathcal{K}_1^{(n)}}_{\mathcal{H}_1},\\ A_n T_1^{(n)} = T_2^{(n)} A_n,\ \kappa^-[I - A_n^{\#} A_n] = \kappa^-[I - A^{\#} A],\\ \kappa^-[I - A_n A_n^{\#}] = \kappa^-[I - AA^{\#}]\}.\end{aligned}$$

For any integers $m \geqslant n \geqslant 0$ define the canonical mapping

$$\varphi_{mn} : \mathcal{S}^{(m)}(A; T_1, T_2) \to \mathcal{S}^{(n)}(A; T_1, T_2)$$

$$\varphi_{mn}(A_m) = P^{\mathcal{K}_2^{(m)}}_{\mathcal{K}_2^{(n)}} A_m \,|\, \mathcal{K}_1^{(n)}, \qquad A_m \in \mathcal{S}^{(m)}(A; T_1, T_2).$$

Then $\{\mathcal{S}^{(n)}(A; T_1, T_2)\}_{n \geqslant 0}$ with the canonical mappings $\{\varphi_{mn}\}_{m \geqslant n \geqslant 0}$ becomes a projective system of sets and consider its projective limit

$$\begin{aligned}\varprojlim_{n \geqslant 0} \mathcal{S}^{(n)}(A; T_1, T_2) = \{(A_n)_{n \geqslant 0} \,|\, A_n \in \mathcal{S}^{(n)}(A; T_1, T_2),\\ A_n = \varphi_{n+1,n}(A_{n+1}),\ n \geqslant 0\}.\end{aligned}$$

In [11, Example 3.4] it is shown that for certain (even finite dimensional) A, T_1, and T_2 it is possible that this projective limit be void. In this section we find a necessary and sufficient condition in order that $\varprojlim_{n \geqslant 0} \mathcal{S}^{(n)}(A; T_1, T_2) \neq \varnothing$ in case when T_1 and T_2 are doubly contractive. For this purpose it is convenient to introduce some more notation. Denote by $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the operator

$$C = AT_1 = T_2 A \tag{6.16}$$

and then the operators $T_r \in \mathcal{L}(\mathcal{H}_1 [+] \mathcal{D}_{T_1^*}, \mathcal{H}_2)$ and $T_c \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2 [+] \mathcal{D}_{T_2})$ by

$$T_r = [AT_1 \quad AD_{T_1^*}], \qquad T_c = [AT_1 \quad D_{T_2} A]'. \tag{6.17}$$

Also, define the operators

$$\begin{aligned}A_1 : \mathcal{D}_{T_1^*} \to \ker D_{C^*}\\ A_1^* = D_{T_1^*} A^* \,|\, \ker D_{C^*}\end{aligned} \tag{6.18}$$

and

$$\begin{aligned} A_2 &: \ker D_C \to \mathscr{D}_{T_2} \\ A_2 &= D_{T_2}A \mid \ker D_C. \end{aligned} \tag{6.19}$$

6.4. THEOREM. *Assume that T_1 and T_2 are doubly contractions, T_r and T_c satisfy the conditions* (3.2) *and* (3.27), *and $\kappa^-[I - A^\# A]$ and $\kappa^-[I - AA^\#]$ are finite. Then the following statements are equivalent*:

(i) *For any $m \geqslant n \geqslant 0$ the canonical mapping φ_{mn} is onto.*

(ii) $\varprojlim_{n \geqslant 0} \mathscr{S}^{(n)}(A; T_1, T_2) \neq \varnothing$.

(iii) $\mathscr{S}^{(1)}(A; T_1, T_2) \neq \varnothing$.

(iv) *The following conditions hold*:

(α) $P^{\mathscr{H}_2}_{\ker D_{C^*}} CJ_1 \mathscr{R}(A_2^*) = \mathscr{R}(A_1)$.

(β) $\kappa^-[I - T_r^\# T_r] = \kappa^-[I - A^\# A]$.

(γ) $\kappa^-[I - T_c T_c^\#] = \kappa^-[I - AA^\#]$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious, while the implication (iii) $\Rightarrow$ (i) follows exactly as in the proof of [11, Theorem 3.5]. In order to prove the equivalence of (iii) with (iv) we recall the method used in the proof of [11, Lemma 3.3]. Just from the definitions it follows that an operator $A_1 \in \mathscr{L}(\mathscr{K}_1^{(1)}, \mathscr{K}_2^{(1)})$ is in $\mathscr{S}^{(1)}(A; T_1, T_2)$ if and only if it is represented by

$$A_1 = \begin{bmatrix} A & 0 \\ X_{11} & X_{12} \end{bmatrix}$$

and

$$X_{11} T_1 + X_{12} D_{T_1} = D_{T_2} A.$$

Considering the operator $B_1 \in \mathscr{L}(\mathscr{H}_1 [+] \mathscr{D}_{T_1^*}, \mathscr{H}_2 [+] \mathscr{D}_{T_2})$

$$B_1 = A_1 R(T_1) \tag{6.20}$$

(recall that $R(T_1)$ is the unitary operator introduced by (2.20)), we obtain

$$B_1 = \begin{bmatrix} C & AD_{T_1^*} \\ D_{T_2} A & S_1 \end{bmatrix}, \tag{6.21}$$

where $S_1 = X_{11} D_{T_1^*} - X_{12} J_{T_1} L_{T_1^*}$. Since $R(T_1)$ is unitary we have

$$\kappa^-[I - B_1^\# B_1] = \kappa^-[I - A_1^\# A_1], \qquad \kappa^-[I - B_1 B_1^\#] = \kappa^-[I - A_1 A_1^\#]$$

and a straightforward argument proves that

$$\kappa^{-}[I-T_r T_r^{\#}]=\kappa^{-}[I-AA^{\#}], \qquad \kappa^{-}[I-T_c^{\#} T_c]=\kappa^{-}[I-A^{\#} A].$$

Hence (6.20) establishes a bijective correspondence between $\mathscr{S}^{(1)}(A; T_1, T_2)$ and the set of solutions of Problem (*) with data T_c, T_r, and $\kappa_1=\kappa^{-}[I-A^{\#}A]$, $\kappa_2=\kappa^{-}[I-AA^{\#}]$. The equivalence of (iii) with (iv) is now a consequence of Theorem 5.1. ∎

6.5. *Remark.* In Theorem 6.4, the condition (γ) is a consequence of (α) and (β). If, in addition, $\mathscr{H}_1$ and $\mathscr{H}_2$ are Pontryagin spaces then (β) and (γ) are always fulfilled.

6.6. *Remark.* If the following identities hold

$$\kappa^{-}[I-C^{\#}C]=\kappa^{-}[I-A^{\#}A], \qquad \kappa^{-}[I-CC^{\#}]=\kappa^{-}[I-AA^{\#}] \quad (6.22)$$

then the conditions (β) and (γ) are fulfilled and the spaces appearing in (α) are both trivial, hence $\underleftarrow{\lim}_{m\geqslant 0}\mathscr{S}^{(n)}(A; T_1, T_2)=\varnothing$, by Theorem 6.4. For the case of T_1 and T_2 doubly contractive, (6.22) are the hypothesis that make [11, Theorem 3.5] work, and in view of [11, Remark 3.6] this is the case of the theorem of lifting of commutants in [23, 25, 10, 13].

6.7. *Remark.* Return to [11, Example 3.4],

$$T_1=\begin{bmatrix}0 & \frac{1}{2}\\ 1 & 0\end{bmatrix}, \qquad T_2=\begin{bmatrix}0 & \sqrt{17}/8\\ 4/\sqrt{17} & 0\end{bmatrix}, \qquad A=\begin{bmatrix}\sqrt{17}/4 & 0\\ 0 & 1\end{bmatrix}.$$

T_1, T_2 are contractions and $\kappa^{-}(I-A^{*}A)=\kappa^{-}(I-AA^{*})=1$.

Moreover, C is a contraction. By a direct computation, $A_1=0$ and $A_2=(\frac{1}{4}\ 0)'$, so that (iv) does not hold. In this case, $\mathscr{S}^{(1)}(A; T_1, T_2)=\varnothing$, as it was directly proved in [11].

Our aim with Theorem 6.4 was to solve some completion problems as those in [1, 16], as their classical variants were solved in [23]. Here we illustrate some aspects concerning two of these problems.

The first one is closely related to the Schur–Takagi problem.

Given a finite dimensional Hilbert space $\mathscr{H}$ and the operators $\{A_k\}_{k=0}^{n} \subset \mathscr{L}(\mathscr{H})$ such that, denoting the Toeplitz block-matrices by

$$S_k=\begin{bmatrix}A_0 & A_1 & \cdots & A_k\\ 0 & A_0 & & \\ \vdots & & \ddots & A_1\\ 0 & 0 & \cdots & A_0\end{bmatrix}, \qquad 0\leqslant k\leqslant n,$$

we have $\kappa^-(I - S_n^* S_n) = \kappa < \infty$, the problem is:

Determine the sequences $\{A_k\}_{k=n+1}^{\infty} \subset \mathscr{L}(\mathscr{H})$ *such that* $\kappa^-(I - S_m^* S_m) = \kappa$, *for any* $m \geqslant n$. (6.23)

6.8. COROLLARY. *Assuming* $n \geqslant 1$, *the Problem* (6.23) *has solutions if and only if*

$$S_{n-1} P_{\ker D_{S_{n-1}}} (A_1^*, A_2^*, ..., A_n^*)^t \mathscr{H} = P_{\ker D_{S_{n-1}^*}} (A_n, A_{n-1}, ..., A_1)^t \mathscr{H}. \tag{6.24}$$

Proof. Following [23] we take $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}^{n+1}$,

$$T_1 = T_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ I & 0 & & & \\ 0 & I & & & \\ \vdots & & \ddots & & \\ 0 & 0 & & I & 0 \end{bmatrix} \in \mathscr{L}(\mathscr{H}^{n+1}),$$

and $A = S_n^*$. Then T_1 and T_2 are contractions, $AT_1 = T_2 A$ and $\kappa^-(I - A^*A) = \kappa < \infty$. Also, we can see that there exists a bijective correspondence between $\varprojlim_{h \geqslant 0} \mathscr{S}^{(k)}(A; T_1, T_2)$ and the set of solutions of Problem (6.23). Now, the conditions (β) and (γ) in Theorem 6.4 are always fulfilled (see Remark 6.5) and the condition (α) reads

$$AT_1 P^{\mathscr{H}_1}_{\ker D_{AT_1}} A^* D_{T_2} \mathscr{H}_2 = P^{\mathscr{H}_2}_{\ker D_{T_1^* A^*}} A D_{T_1^*} \mathscr{H}_1. \tag{6.25}$$

It is easy to verify that, for the Problem (6.23), the conditions (6.24) and (6.25) are equivalent. ∎

6.9. *Remark.* In the scalar case (i.e., $\mathscr{H} = \mathbb{C}$) we obtain that for $n = 0$ (direct argument) and $n = 1$ (using Corollary 6.8) the Problem (6.23) always has solutions. However, for $n = 2$ this is not the case, as we can see taking $A_0 = A_2 = 1$ and $A_1 = 0$ (of course, we have to use Corollary 6.8).

The second application is closely connected with the Nehari problem. Given a finite dimensional Hilbert space $\mathscr{H}$, let $\{C_{-k}\}_{k \geqslant 1} \subset \mathscr{L}(\mathscr{H})$ be a sequence of operators such that the Hankel block-matrix

$$H_{-1} = \begin{bmatrix} C_{-1} & C_{-2} & C_{-3} & \cdots \\ C_{-2} & C_{-3} & \cdots & \\ C_{-3} & \cdots & & \\ \vdots & & & \end{bmatrix}$$

defines a bounded operator on $l^2(\mathbb{N}) \otimes \mathscr{H}$ and $\kappa^-(I - H_{-1}^* H_{-1}) = \kappa < \infty$.

The problem is

Determine the sequences $\{C_k\}_{k\geqslant 0}\subset \mathscr{L}(\mathscr{H})$ *such that* $\kappa^-(I-H_k^*H_k)=\kappa$, *for any* $k\geqslant 0$, (6.26)

where we have denoted

$$H_k=\begin{bmatrix} C_k & C_{k-1} & C_{k-2} & \cdots \\ C_{k-1} & C_{k-2} & \cdots & \\ C_{k-2} & \cdots & & \\ \vdots & & & \end{bmatrix}, \qquad k\in\mathbb{Z}.$$

6.10. COROLLARY. *The Problem* (6.26) *has solutions if and only if*

$$H_{-2}P_{\ker D_{H_{-2}}}(C_{-1}^*, C_{-2}^*, ...)^t\mathscr{H}=P_{\ker D_{H_{-2}^*}}(C_{-1}, C_{-2}, ...)^t. \tag{6.27}$$

Proof. We take $\mathscr{H}_1=\mathscr{H}_2=1^2(\mathbb{N})\otimes\mathscr{H}$, T_1 is the left shift of multiplicity $\mathscr{H}$, $T_2=T_1^*$, and $A=H_{-1}$. T_1 and T_2 are contractions and $AT_1=T_2A=H_{-2}$. The rest of the proof is similar with that of Corollary 6.8. ∎

As problem (6.23) is a particular case of (6.26) (take $C_{-1}=A_n, ...,$ $C_{-n-1}=A_0$), the example exhibited in Remark 6.9 shows that (6.27) is also consistent.

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